

Trudinger-Moser inequalities on complete noncompact Riemannian manifolds

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Abstract

Let (M, g) be a complete noncompact Riemannian n -manifold ($n \geq 2$). If there exist positive constants α , τ and β such that

$$\sup_{u \in W^{1,n}(M), \|u\|_{1,\tau} \leq 1} \int_M \left(e^{\alpha|u|^{\frac{n}{n-1}}} - \sum_{k=0}^{n-2} \frac{\alpha^k |u|^{\frac{nk}{n-1}}}{k!} \right) dv_g \leq \beta,$$

where $\|u\|_{1,\tau} = \|\nabla_g u\|_{L^n(M)} + \tau \|u\|_{L^n(M)}$, then we say that Trudinger-Moser inequality holds. Suppose Trudinger-Moser inequality holds, we prove that there exists some positive constant ϵ such that $\text{Vol}_g(B_x(1)) \geq \epsilon$ for all $x \in M$. Also we give a sufficient condition under which Trudinger-Moser inequality holds, say the Ricci curvature of (M, g) has lower bound and its injectivity radius is positive. Moreover, Adams inequality is discussed in this paper. For application of Trudinger-Moser inequalities, we obtain existence results for some quasilinear equations with nonlinearity of exponential growth.

Key words: Trudinger-Moser inequality, Adams inequality, exponential growth

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1. Introduction

Let Ω be a smooth bounded domain in \mathbb{R}^n ($n \geq 2$) and $C_0^\infty(\Omega)$ be a space of smooth functions with compact support in Ω . Let $W_0^{m,p}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ under the Sobolev norm

$$\|u\|_{W_0^{m,p}(\Omega)} := \left(\sum_{l=0}^m \int_\Omega |\nabla^l u|^p dx \right)^{1/p}. \quad (1.1)$$

Assume that m is an integer satisfying $1 \leq m < n$. Then Sobolev embedding theorem asserts that $W_0^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$, $1 \leq q \leq np/(n-mp)$. Concerning the limiting case $mp = n$, one has $W_0^{m,n/m}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q \geq 1$. But the embedding is not valid for $q = \infty$. To fill this gap, it is natural to find the maximal growth function $g : \mathbb{R} \rightarrow \mathbb{R}^+$ such that

$$\sup_{u \in W_0^{m,n/m}(\Omega), \|u\|_{W_0^{m,n/m}(\Omega)} \leq 1} \int_\Omega g(u) dx < \infty.$$

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In the case $m = 1$, Trudinger [38] and Pohozaev [33] found independently that the maximal growth is of exponential type. More precisely, there exist two positive constants α_0 and C depending only on n such that

$$\sup_{u \in W_0^{1,n}(\Omega), \|u\|_{W_0^{1,n}(\Omega)} \leq 1} \int_{\Omega} e^{\alpha_0 |u|^{\frac{n}{n-1}}} dx \leq C|\Omega|, \quad (1.2)$$

where $|\Omega|$ denotes the Lebesgue measure of Ω . Moser [30] obtained the best constant $\alpha_n = n\omega_{n-1}^{1/(n-1)}$ such that the above supremum is finite when α_0 is replaced by α_n , where ω_{n-1} is the area of the unit sphere in \mathbb{R}^n . Moser's work relies on a rearrangement argument [17]. In literature the kind of inequalities like (1.2) are called Trudinger-Moser inequalities.

Adams [2] generalized inequality (1.2) to the case of general $m : 1 \leq m < n$ as follows. For any $u \in W_0^{m,n/m}(\Omega)$, the l -th order gradient of u reads

$$\nabla^l u = \begin{cases} \Delta^{\frac{l}{2}} u, & \text{if } l \text{ is even,} \\ \nabla \Delta^{\frac{l-1}{2}} u, & \text{if } l \text{ is odd,} \end{cases} \quad (1.3)$$

there exists a positive constant $C_{m,n}$ such that

$$\sup_{u \in W_0^{m,n/m}(\Omega), \|u\|_{W_0^{m,n/m}(\Omega)} \leq 1} \int_{\Omega} e^{\beta_0 |u|^{\frac{n}{n-m}}} dx \leq C_{m,n} |\Omega|, \quad (1.4)$$

where β_0 is the best constant depending only on n and m , namely

$$\beta_0 = \beta_0(m,n) := \begin{cases} \frac{n}{\omega_{n-1}} \left[\frac{\pi^{n/2} 2^m \Gamma((m+1)/2)}{\Gamma((n-m+1)/2)} \right] & \text{when } m \text{ is odd} \\ \frac{n}{\omega_{n-1}} \left[\frac{\pi^{n/2} 2^m \Gamma(m/2)}{\Gamma((n-m)/2)} \right] & \text{when } m \text{ is even.} \end{cases} \quad (1.5)$$

The inequality (1.4) is known as Adams inequality. Adams first represented a function u in terms of its gradient function $\nabla^m u$ by using a convolution operator. Then using the O'Neil's idea [31] of rearrangement of convolution of two functions and the idea which originally goes back to Garcia, he obtained (1.4).

There are many types of extensions for Trudinger-Moser inequality and Adams inequality. One is to establish such inequalities on the whole euclidian space \mathbb{R}^n . Cao [8] employed the decreasing rearrangement argument to prove that for all $\alpha < 4\pi$ and $A > 0$, there exists a constant C depending only on α and A such that for all $u \in W^{1,2}(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2} |\nabla u|^2 dx \leq 1$, $\int_{\mathbb{R}^2} u^2 dx \leq A$, there holds

$$\int_{\mathbb{R}^2} (e^{\alpha u^2} - 1) dx \leq C. \quad (1.6)$$

His argument was generalized to n -dimensional case by do Ó [12] and Panda [32] independently. Later, Adachi-Tanaka [1] gave another type of generalization. All these inequalities are subcritical ones since $\alpha < \alpha_n$. It was Ruf [35] who first proved the critical Trudinger-Moser inequality in the whole euclidian space \mathbb{R}^2 and gave out extremal functions via more delicate analysis. This result was generalized to n -dimensional case by Li-Ruf [25] through combining symmetrization and blow-up analysis. Subsequently, using the decreasing rearrangement argument and Young's inequality, Adimurthi-Yang [4] derived an interpolation of Trudinger-Moser inequality and Hardy inequality in \mathbb{R}^n , which can be viewed as a singular Trudinger-Moser inequality.

Another kind of singular Trudinger-Moser inequality was recently established by Wang-Ye [39] through the method of blow-up analysis.

Substantial progresses on Adams inequality in \mathbb{R}^n was also made recently. Following lines of Adams, Kozono et al. [19] obtained subcritical Adams inequality in the whole euclidian space \mathbb{R}^n . Based on rearrangement argument of Trombetti-Vazquez [37], Ruf-Sani [36] proved the critical Adams inequalities in \mathbb{R}^n as follows. Let m be an even integer less than n . Assume that $u \in W_0^{m,n/m}(\mathbb{R}^n)$ and $\|(-\Delta + I)^{m/2}u\|_{L^{n/m}(\mathbb{R}^n)} \leq 1$. There exists a constant $C > 0$ depending only on n and m such that

$$\int_{\mathbb{R}^n} \left(e^{\beta_0 |u|^{\frac{n}{n-m}}} - \sum_{k=0}^{j-2} \frac{\beta_0^k |u|^{\frac{nk}{n-m}}}{k!} \right) dx < C,$$

where j is the smallest integer great than or equal to n/m .

Another extension is to establish Trudinger-Moser inequality and Adams inequality on compact Riemannian manifolds. Let (M, g) be a compact Riemannian n -manifold. For $u \in W^{1,n}(M)$, it was shown by Aubin [5] that $\exp(\alpha |u|^{n/(n-1)} \|u\|_{W^{1,n}(M)}^{-n/(n-1)})$ is integrable for sufficiently small $\alpha > 0$ which does not depend on u . In fact, this is an easy consequence of Trudinger-Moser inequality and finite partition of unity on M . Let $\tilde{\alpha}$ be the supremum of the above α 's. It was first found by Cherrier [9] that $\tilde{\alpha} = \alpha_n$. Cherrier [10] obtained similar results for $u \in W_{m,n/m}(M)$. Following the lines of Adams, Fontana [15] obtained critical Adams inequality on (M, g) . In 1997, using the method of blow-up analysis, Ding et al. [11] established a nice Trudinger-Moser inequality on compact Riemannian surface and successfully applied it to deal with the prescribed Gaussian curvature problem. Adapting the argument of Ding et al., Li [21, 22] and Li-Liu [23] proved the existence of extremal functions for Trudinger-Moser inequalities. Their idea was also employed by the author [40, 41, 42] to find extremal functions for various Trudinger-Moser type inequalities. For vector bundles over a compact Riemannian 2-manifold, Li-Liu-Yang obtained Trudinger-Moser inequalities in [24].

Among other contributions, we mention the following results. Using the method of blow-up analysis, Adimurthi-Druet [3] proved that when $0 \leq \alpha < \lambda_1(\Omega)$, there holds

$$\sup_{u \in W_0^{1,2}(\Omega), \|\nabla u\|_2 \leq 1} \int_{\Omega} e^{4\pi u^2(1+\alpha\|u\|_2^2)} dx < \infty,$$

where $\lambda_1(\Omega)$ is the first eigenvalue of Laplacian on bounded smooth domain $\Omega \subset \mathbb{R}^2$. Moreover, the supremum is infinite when $\alpha \geq \lambda_1(\Omega)$. Later this result was generalized by the author [43] and Lu-Yang [27, 28, 29].

Although there are fruitful results on euclidian space and compact Riemannian manifolds, we know little about Trudinger-Moser inequalities on complete noncompact Riemannian manifolds. In this paper, we concern this problem. Let (M, g) be any complete noncompact Riemannian n -manifold. Throughout this paper, all the manifolds are assumed to be without boundary, and of dimension $n \geq 2$. We say that Trudinger-Moser inequality holds on (M, g) if there exist positive constants α , τ and β such that

$$\sup_{u \in W^{1,n}(M), \|u\|_{1,\tau} \leq 1} \int_M \left(e^{\alpha |u|^{\frac{n}{n-1}}} - \sum_{k=0}^{n-2} \frac{\alpha^k |u|^{\frac{nk}{n-1}}}{k!} \right) dv_g \leq \beta, \quad (1.7)$$

where

$$\|u\|_{1,\tau} = \left(\int_M |\nabla_g u|^n dv_g \right)^{1/n} + \tau \left(\int_M |u|^n dv_g \right)^{1/n}. \quad (1.8)$$

If the above supremum is infinite for all $\alpha > 0$ and $\tau > 0$, then we say that Trudinger-Moser inequality is not valid on (M, g) . Motivated by Sobolev embedding (Hebey [18], Chapter 3), in this paper, we propose and answer the following three questions.

(Q_1) Which kind of complete noncompact Riemannian manifolds can possibly make Trudinger-Moser inequalities hold?

(Q_2) What geometric assumptions should we consider in order to obtain Trudinger-Moser inequalities on complete noncompact Riemannian manifolds?

(Q_3) Are those geometric assumptions in (Q_2) necessary?

This paper is organized as follows: In Section 2, we state our main results. From section 3 to section 5, we answer the questions (Q_1)-(Q_3), respectively. Adams inequalities are considered in section 6. Finally, Trudinger-Moser inequalities are applied to nonlinear analysis in section 7.

2. Main results

In this section, we answer questions (Q_1)-(Q_3), and give an application of Trudinger-Moser inequality. Throughout this paper, we denote for simplicity a function $\zeta : \mathbb{N} \times [0, \infty) \rightarrow \mathbb{R}$ by

$$\zeta(l, t) = e^t - \sum_{k=0}^{l-2} \frac{t^k}{k!}, \quad \forall l \geq 2. \quad (2.1)$$

From ([44], lemma 2.1 and lemma 2.2), we know that

$$(\zeta(l, t))^q \leq \zeta(l, qt) \quad (2.2)$$

and

$$\zeta(l, t) \leq \frac{1}{\mu} \zeta(l, \mu t) + \frac{1}{\nu} \zeta(l, \nu t). \quad (2.3)$$

for all $l \geq 2$, $q \geq 1$, $t \in [0, \infty)$, and $\mu > 0$, $\nu > 0$ satisfying $1/\mu + 1/\nu = 1$.

The following proposition answers question (Q_1).

Proposition 2.1. *Let (M, g) be a complete Riemannian n -manifold. Suppose that Trudinger-Moser inequality holds on (M, g) , i.e. there exist positive constants α , τ and β such that (1.7) holds. Then the Sobolev space $W^{1,n}(M)$ is embedded in $L^q(M)$ continuously for any $q \geq n$. Furthermore, for any $r > 0$ there exists a positive constant ϵ depending only on n , α , τ , β and r such that $\text{Vol}_g(B_x(r)) \geq \epsilon$ for all $x \in M$, where $B_x(r)$ denotes the geodesic ball centered at x with radius r .*

From proposition 2.1 we know that there are indeed complete noncompact Riemannian manifolds such that Trudinger-Moser inequalities are not valid, namely

Corollary 2.2. *For any integer $n \geq 2$, there exists a complete noncompact Riemannian n -manifold on which Trudinger-Moser inequality is not valid.*

To answer question (Q_2), we have the following:

Theorem 2.3. *Let (M, g) be a complete noncompact Riemannian n -manifold. Suppose that its Ricci curvature has lower bound, namely $\text{Rc}_{(M,g)} \geq Kg$ for some constant $K \in \mathbb{R}$, and its injectivity radius is strictly positive, namely $\text{inj}_{(M,g)} \geq i_0$ for some constant $i_0 > 0$. Then we have*

- (i) *for any $0 \leq \alpha < \alpha_n = n\omega_{n-1}^{1/(n-1)}$, there exists positive constants τ and β depending only on n, α, K and i_0 such that (1.7) holds. As a consequence, $W^{1,n}(M)$ is embedded in $L^q(M)$ continuously for any $q \geq n$;*
- (ii) *for any $\alpha > \alpha_n$ and any $\tau > 0$, the supremum in (1.7) is infinite;*
- (iii) *for any $\alpha > 0$ and any $u \in W^{1,n}(M)$, there holds $\zeta(n, \alpha|u|^{n/(n-1)}) \in L^1(M)$.*

Now we turn to question (Q_3). The following proposition implies that one of the hypotheses of theorem 2.3, the injectivity radius is strictly positive, can not be removed.

Proposition 2.4. *For any integer $n \geq 2$, there exists a complete noncompact Riemannian n -manifold, whose Ricci curvature has lower bound, such that Trudinger-Moser inequality is not valid on it.*

We shall construct complete noncompact Riemannian manifolds on which Trudinger-Moser inequalities hold, but their Ricci curvatures are unbounded from below. This implies that the other hypothesis of theorem 2.3, Ricci curvature has lower bound, is not necessary. Namely

Proposition 2.5. *For any integer $n \geq 2$, there exists a complete noncompact Riemannian n -manifold on which Trudinger-Moser inequality holds, but its Ricci curvature is unbounded from below.*

Let us explain the idea of proving proposition 2.1 and theorem 2.3. The first part of conclusions of proposition 2.1, $W^{1,n}(M) \hookrightarrow L^q(M)$ for all $q \geq n$, is based on an observation

$$\int_M \zeta\left(n, \alpha|u|^{\frac{n}{n-1}}\right) dv_g = \sum_{k=n-1}^{\infty} \frac{\alpha^k}{k!} \int_M |u|^{\frac{nk}{n-1}} dv_g.$$

To find some $\epsilon > 0$ such that $\text{Vol}_g(B_x(r)) \geq \epsilon$ for all $x \in M$, we employ the method of Carron ([18], lemma 3.2) who obtained similar result for Sobolev embedding. For the proof of theorem 2.3, we first derive a uniform local Trudinger-Moser inequality (lemma 4.2 below). Then using harmonic coordinates and Gromov's covering lemma, we get the desired global Trudinger-Moser inequality. The proofs of corollary 2.2, proposition 2.4 and proposition 2.5 are all based on construction of Riemannian manifolds.

Concerning Adams inequalities on complete noncompact Riemannian manifolds, we have the following:

Theorem 2.6. *Let (M, g) be a complete noncompact Riemannian n -manifold. Suppose that there exist positive constants $C(k)$ and i_0 such that $|\nabla_g^k \text{Rc}_{(M,g)}| \leq C(k)$, $k = 0, 1, \dots, m-1$, $\text{inj}_{(M,g)} \geq i_0 > 0$. Let $j = n/m$ when n/m is an integer, and $j = [n/m] + 1$ when n/m is not an integer, where $[n/m]$ denotes the integer part of n/m . Then we conclude the following:*

(i) there exist positive constants α_0 and β depending only on $n, m, C(k), k = 1, \dots, m-1$, and i_0 such that

$$\sup_{\|u\|_{W^{m,n/m}(M)} \leq 1} \int_M \zeta\left(j, \alpha_0 |u|^{\frac{n}{n-m}}\right) dv_g \leq \beta.$$

As a consequence, $W^{m,n/m}(M)$ is embedded in $L^q(M)$ continuously for any $q \geq n/m$;

(ii) for any $\alpha > 0$ and any $u \in W^{m,n/m}(M)$, there holds $\zeta(j, \alpha |u|^{n/(n-m)}) \in L^1(M)$.

The proof of theorem 2.6 is similar to that of theorem 2.3. It should be remarked that the existing proofs of Trudinger-Moser inequalities or Adams inequalities for the euclidian space \mathbb{R}^n are all based on rearrangement argument, which is difficult to be applied to complete noncompact Riemannian manifold case. Our method is from uniform local estimates to global estimates. It does not depend on the rearrangement theory directly.

Trudinger-Moser inequality plays an important role in nonlinear analysis. Let (M, g) be a complete noncompact Riemannian n -manifold. ∇_g denotes its covariant derivative, and div_g denotes its divergence operator. Assume the Ricci curvature of (M, g) has lower bound and the injectivity radius is strictly positive. We consider the existence results for the following quasilinear equation.

$$-\text{div}_g(|\nabla_g u|^{n-2} \nabla_g u) + v(x)|u|^{n-2}u = \phi(x)f(x, u), \quad (2.4)$$

where $v(x)$, $\phi(x)$ and $f(x, t)$ are all continuous functions, and $f(x, t)$ behaves like $e^{\alpha t^{n/(n-1)}}$ as $t \rightarrow +\infty$. In the case that (M, g) is the standard euclidean space \mathbb{R}^n and $\phi(x) = |x|^{-\beta}$ ($0 \leq \beta < n$), problem (2.4) has been studied by do Ó et. al. [13, 14], Adimurthi-Yang [4], Yang [44], Lam-Lu [20] and Zhao [45]. Let O be a fixed point of M and $d_g(\cdot, \cdot)$ be the geodesic distance between two points of (M, g) . Assume that $\phi(x)$ satisfies the following hypotheses.

- (ϕ_1) $\phi(x) \in L^p_{\text{loc}}(M)$ for some $p > 1$, i. e., for any $R > 0$ there holds $\phi(x) \in L^p(B_O(R))$;
- (ϕ_2) $\phi(x) > 0$ for all $x \in M$ and there exist positive constants C_0 and R_0 such that $\phi(x) \leq C_0$ for all $x \in M \setminus B_O(R_0)$.

The potential $v(x)$ is assumed to satisfy the following:

- (v_1) there exists some constant $v_0 > 0$ such that $v(x) \geq v_0$ for all $x \in M$;
- (v_2) either $v(x) \in L^{1/(n-1)}(M)$ or $v(x) \rightarrow +\infty$ as $d_g(O, x) \rightarrow +\infty$.

The nonlinearity $f(x, t)$ satisfies the following hypotheses.

- (f_1) there exist constants $\alpha_0, b_1, b_2 > 0$ such that for all $(x, t) \in M \times \mathbb{R}^+$,

$$|f(x, t)| \leq b_1 t^{n-1} + b_2 \zeta\left(n, \alpha_0 t^{n/(n-1)}\right);$$

- (f_2) there exists some constant $\mu > n$ such that for all $x \in M$ and $t > 0$,

$$0 < \mu F(x, t) \equiv \mu \int_0^t f(x, s) ds \leq t f(x, t);$$

(f_3) there exist constants $R_1, A_1 > 0$ such that if $t \geq R_1$, then for all $x \in M$ there holds

$$F(x, t) \leq A_1 f(x, t).$$

Define a function space

$$E = \left\{ u \in W^{1,n}(M) : \int_M v(x)|u|^n dv_g < \infty \right\}. \quad (2.5)$$

We say that $u \in E$ is a weak solution of problem (2.4) if for all $\varphi \in E$ we have

$$\int_M (|\nabla_g u|^{n-2} \nabla_g u \nabla_g \varphi + v(x)|u|^{n-2} u \varphi) dv_g = \int_M \phi(x) f(x, u) \varphi dv_g.$$

Define a weighted eigenvalue for the n -Laplace operator by

$$\lambda_\phi = \inf_{u \in E, u \neq 0} \frac{\int_M (|\nabla_g u|^n + v(x)|u|^n) dv_g}{\int_M \phi(x)|u|^n dv_g}. \quad (2.6)$$

Then we state the following:

Theorem 2.7. *Let (M, g) be a complete noncompact Riemannian n -manifold. Suppose that $\text{Rc}_{(M,g)} \geq Kg$ for some constant $K \in \mathbb{R}$, and $\text{inj}_{(M,g)} \geq i_0$ for some positive constant i_0 . Assume that $v(x)$ is a continuous function satisfying (v_1) and (v_2) , $\phi(x)$ is a continuous function satisfying (ϕ_1) and (ϕ_2) , $f : M \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and the hypotheses (f_1) , (f_2) and (f_3) are satisfied. Furthermore we assume*

(f_4) $\limsup_{t \rightarrow 0^+} nF(x, t)/t^n < \lambda_\phi$ uniformly in $x \in M$;

(f_5) there exist constants $q > n$ and C_q such that for all $(x, t) \in M \times [0, \infty)$

$$f(x, t) \geq C_q t^{q-1},$$

where

$$C_q > \left(\frac{q-n}{q} \right)^{(q-n)/n} \left(\frac{p\alpha_0}{(p-1)\alpha_n} \right)^{(q-n)(n-1)/n} S_q^q$$

and

$$S_q = \inf_{u \in E \setminus \{0\}} \frac{\left(\int_M (|\nabla_g u|^n + v(x)|u|^n) dv_g \right)^{1/n}}{\left(\int_M \phi(x)|u|^q dv_g \right)^{1/q}}. \quad (2.7)$$

Then the problem (2.4) has a nontrivial nonnegative weak solution.

Remark 2.8. We shall prove that S_q can be attained (lemma 7.2 below). When (M, g) is the standard euclidian space \mathbb{R}^n , $\phi(x) = |x|^{-\beta}$ for $0 \leq \beta < n$, (f_1) -(f_4) and

$$(H_5) \liminf_{s \rightarrow +\infty} s f(x, s) e^{-\alpha_0 s^{\frac{n}{n-1}}} = \beta_0 > M$$

uniformly in x , where M is some sufficiently large number, we obtained similar existence result in [44]. The following proposition implies that the set of functions satisfying (f_1) -(f_5) is not empty and assumptions (f_1) -(f_5) do not imply (H_5) .

Proposition 2.9. *There exist continuous functions $f : M \times \mathbb{R} \rightarrow \mathbb{R}$ such that (f_1) – (f_5) are satisfied, but (H_5) is not satisfied.*

We also consider multiplicity results for a perturbation of the problem (2.4), namely

$$-\operatorname{div}_g(|\nabla_g u|^{n-2} \nabla_g u) + v(x)|u|^{n-2}u = \phi(x)f(x, u) + \epsilon h(x), \quad (2.8)$$

where $h(x) \in E^*$, the dual space of E . If $h \not\equiv 0$ and $\epsilon > 0$ is sufficiently small, under some assumptions there exist at least two distinct weak solutions to (2.8). Precisely, we have the following theorem.

Theorem 2.10. *Let (M, g) be a complete noncompact Riemannian n -manifold. Suppose that $\operatorname{Rc}_{(M,g)} \geq Kg$ for some constant $K \in \mathbb{R}$, and $\operatorname{inj}_{(M,g)} \geq i_0$ for some positive constant i_0 . Assume $f(x, t)$ is continuous in $M \times \mathbb{R}$ and (f_1) – (f_5) are satisfied. Both $v(x)$ and $\phi(x)$ are continuous in M and (v_1) , (v_2) , (ϕ_1) , (ϕ_2) are satisfied, h belongs to E^* , the dual space of E , with $h \geq 0$ and $h \not\equiv 0$. Then there exists $\epsilon_0 > 0$ such that if $0 < \epsilon < \epsilon_0$, then the problem (2.8) has at least two distinct nonnegative weak solutions.*

The proofs of theorem 2.7 and theorem 2.10 are based on theorem 2.3, Mountain-pass theorem and Ekeland's variational principle. Though similar idea was used in the case (M, g) is the standard euclidian space \mathbb{R}^n [4, 13, 14, 20, 44], technical difficulties caused by manifold structure must be smoothed.

3. Necessary conditions

In this section, we consider the necessary conditions under which Trudinger-Moser inequality holds. Precisely we shall prove proposition 2.1 and corollary 2.2. Firstly we have the following:

Lemma 3.1. *Let (M, g) be a complete Riemannian n -manifold. Suppose that there exist constants $q > n$, $A > 0$ and $\tau > 0$ such that for all $u \in W^{1,n}(M)$, there holds*

$$\left(\int_M |u|^q dv_g \right)^{1/q} \leq A \|u\|_{1,\tau}, \quad (3.1)$$

where $\|u\|_{1,\tau}$ is defined by (1.8). Then for any $r > 0$ there exists some positive constant ϵ depending only on A , n , q , τ , and r such that for all $x \in M$, $\operatorname{Vol}_g(B_x(r)) \geq \epsilon$.

Proof. Let $r > 0$, $x \in M$, and $\phi \in W^{1,n}(M)$ be such that $\phi = 0$ on $M \setminus B_x(r)$. By Hölder's inequality,

$$\left(\int_M |\phi|^n dv_g \right)^{1/n} \leq \operatorname{Vol}_g(B_x(r))^{\frac{1}{n} - \frac{1}{q}} \left(\int_M |\phi|^q dv_g \right)^{1/q}.$$

This together with (3.1) gives

$$\left(1 - \tau A \operatorname{Vol}_g(B_x(r))^{\frac{1}{n} - \frac{1}{q}} \right) \left(\int_M |\phi|^q dv_g \right)^{1/q} \leq A \left(\int_M (|\nabla \phi|^n dv_g) \right)^{1/n}. \quad (3.2)$$

Fix $x \in M$ and $R > 0$. Then either

$$\text{Vol}_g(B_x(R)) > \left(\frac{1}{2\tau A} \right)^{nq/(q-n)} \quad (3.3)$$

or

$$\text{Vol}_g(B_x(R)) \leq \left(\frac{1}{2\tau A} \right)^{nq/(q-n)}. \quad (3.4)$$

If (3.4) holds, then we have

$$1 - \tau A \text{Vol}_g(B_x(R))^{\frac{1}{n} - \frac{1}{q}} \geq 1/2,$$

and whence for all $r \in (0, R]$ and all $\phi \in W^{1,n}(M)$ with $\phi = 0$ on $M \setminus B_x(r)$,

$$\left(\int_M |\phi|^q dv_g \right)^{1/q} \leq 2A \left(\int_M (|\nabla \phi|^n dv_g) \right)^{1/n}. \quad (3.5)$$

Now we set

$$\phi(y) = \begin{cases} r - d_g(x, y) & \text{when } d_g(x, y) \leq r \\ 0 & \text{when } d_g(x, y) > r. \end{cases}$$

Clearly $\phi \in W^{1,n}(M)$, $\phi = 0$ on $M \setminus B_x(r)$, $\phi \geq r/2$ on $B_x(r/2)$, and $|\nabla \phi| = 1$ almost everywhere in $B_x(r)$. It then follows from (3.5) that

$$\frac{r}{2} \text{Vol}_g(B_x(r/2))^{1/q} \leq 2A \text{Vol}_g(B_x(r))^{1/n}.$$

Hence we have for all $r \leq R$,

$$\text{Vol}_g(B_x(r)) \geq \left(\frac{r}{4A} \right)^n \text{Vol}_g(B_x(r/2))^{n/q}.$$

By induction we obtain for any positive integer m ,

$$\text{Vol}_g(B_x(R)) \geq \left(\frac{R}{2A} \right)^{n\alpha(m)} \left(\frac{1}{2} \right)^{n\beta(m)} \text{Vol}_g(B_x(R/2^m))^{(n/q)^m}, \quad (3.6)$$

where

$$\alpha(m) = \sum_{j=1}^m (n/q)^{j-1}, \quad \beta(m) = \sum_{j=1}^m j(n/q)^{j-1}.$$

On one hand we know from ([7], Theorem 3.98) that $\text{Vol}_g(B_x(r)) = \frac{\omega_{n-1}}{n} r^n (1 + o(r))$, where ω_{n-1} is the area of the euclidean unit sphere in \mathbb{R}^n , and $o(r) \rightarrow 0$ as $r \rightarrow 0$. One can see without any difficulty that

$$\lim_{m \rightarrow \infty} \text{Vol}_g(B_x(R/2^m))^{(n/q)^m} = 1.$$

On the other hand we have

$$\sum_{j=1}^{\infty} (n/q)^{j-1} = \frac{q}{q-n}, \quad \sum_{j=1}^{\infty} j(n/q)^{j-1} = \frac{q^2}{(q-n)^2}.$$

Hence, passing to the limit $m \rightarrow \infty$ in (3.6), one concludes that

$$\text{Vol}_g(B_x(R)) \geq \left(\frac{R}{2^{(2q-n)/(q-n)} A} \right)^{nq/(q-n)}.$$

This together with (3.3), (3.4) implies that

$$\text{Vol}_g(B_x(R)) \geq \min \left\{ \frac{1}{2\tau A}, \frac{R}{2^{(2q-n)/(q-n)} A} \right\}^{nq/(q-n)}$$

and completes the proof of the lemma. \square

It should be pointed out that the above argument is a modification of that of Carron ([18], lemma 3.2). Note that the condition (3.1) implies that $W^{1,n}(M)$ is continuously embedded in $L^q(M)$ for some $q > n$. This is different from the assumption of ([18], lemma 3.2).

To prove proposition 2.1, we also need the following interpolation inequality.

Lemma 3.2. *Let τ be any positive real number. Suppose there exist positive constants q_1, q_2, A_1 and A_2 such that $q_2 > q_1 > 0$ and*

$$\left(\int_M |u|^{q_i} dv_g \right)^{1/q_i} \leq A_i \|u\|_{1,\tau} \quad (3.7)$$

for all $u \in W^{1,n}(M)$, $i = 1, 2$. Then for all $q : q_1 < q < q_2$ there exists a positive constant $A = A(A_1, A_2, q_1, q_2)$ such that

$$\left(\int_M |u|^q dv_g \right)^{1/q} \leq A \|u\|_{1,\tau} \quad (3.8)$$

for all $u \in W^{1,n}(M)$.

Proof. For any $u \in W^{1,n}(M) \setminus \{0\}$, we set $\tilde{u} = u/\|u\|_{1,\tau}$. It follows from (3.7) that

$$\left(\int_M |\tilde{u}|^{q_i} dv_g \right)^{1/q_i} \leq A_i, \quad i = 1, 2.$$

Assume $q_1 < q < q_2$. Since $|\tilde{u}|^q \leq |\tilde{u}|^{q_1} + |\tilde{u}|^{q_2}$, there holds

$$\int_M |\tilde{u}|^q dv_g \leq \int_M |\tilde{u}|^{q_1} dv_g + \int_M |\tilde{u}|^{q_2} dv_g \leq A_1^{q_1} + A_2^{q_2}.$$

Hence

$$\left(\int_M |u|^q dv_g \right)^{1/q} \leq (A_1^{q_1} + A_2^{q_2})^{\frac{1}{q}} \|u\|_{1,\tau}.$$

Take $A = \max\{(A_1^{q_1} + A_2^{q_2})^{1/q_1}, (A_1^{q_1} + A_2^{q_2})^{1/q_2}\}$. Then (3.8) follows immediately. \square

Proof of proposition 2.1. Assume there exist positive constants α , τ and β such that (1.7) holds. For any $u \in W^{1,n}(M)$ we set $\tilde{u} = u/\|u\|_{1,\tau}$. It follows from (1.7) that

$$\int_M \sum_{k=n-1}^{\infty} \frac{\alpha^k |\tilde{u}|^{\frac{nk}{n-1}}}{k!} dv_g \leq \beta.$$

Particularly for any integer $k \geq n-1$ there holds

$$\int_M \frac{\alpha^k |\tilde{u}|^{\frac{nk}{n-1}}}{k!} dv_g \leq \beta,$$

and thus

$$\left(\int_M |\tilde{u}|^{\frac{nk}{n-1}} dv_g \right)^{\frac{n-1}{nk}} \leq \left(\frac{k! \beta}{\alpha^k} \right)^{\frac{n-1}{nk}} \|\tilde{u}\|_{1,\tau}.$$

For any $q \geq n$, there exists some $k \geq n-1$ such that

$$\frac{nk}{n-1} \leq q < \frac{n(k+1)}{n-1}.$$

In fact we can choose $k = [(n-1)p/n]$, the integer part of $(n-1)p/n$. By lemma 3.2, there exists a positive constant A depending only on n , q , α , and β such that

$$\left(\int_M |u|^q dv_g \right)^{1/q} \leq A \|u\|_{1,\tau}.$$

This implies that $W^{1,n}(M) \hookrightarrow L^q(M)$ continuously. Now we fix some $q > n$, say $q = n+1$. Then by lemma 3.1, there exists some constant $\epsilon > 0$ depending only on n , α , τ , β and r such that for all $x \in M$, $\text{Vol}_g(B_x(r)) \geq \epsilon$. \square

Proof of corollary 2.2. For any complete noncompact Riemannian n -manifold (M, g) , if Trudinger-Moser inequality holds, then by proposition 2.1, there exists some constant $\epsilon > 0$ such that $\text{Vol}_g(B_x(r)) \geq \epsilon$ for all $x \in M$. Hence if there exists some complete noncompact Riemannian n -manifold (M, g) such that

$$\inf_{x \in M} \text{Vol}_g(B_x(r)) = 0,$$

then we conclude that Trudinger-Moser inequality is not valid on it. Now we construct such complete Riemannian manifolds. Consider the warped product

$$M = \mathbb{R} \times N, \quad g(t, \theta) = dt^2 + f(t) ds_N^2,$$

where (N, ds_N^2) is a compact $(n-1)$ -Riemannian manifold, dt^2 is the euclidian metric of \mathbb{R} , and f is a smooth function satisfying $f(t) > 0, \forall t \in \mathbb{R}$ and $\lim_{t \rightarrow +\infty} f(t) = 0$. If $y = (t_1, m_1)$ and $z = (t_2, m_2)$ are two points of M , then $d_g(y, z) \geq |t_2 - t_1|$. This together with the compactness of N implies that (M, g) is complete. In addition, for any $x = (t, m) \in M$, there holds

$$B_x(1) \subset (t-1, t+1) \times N.$$

Therefore

$$\begin{aligned}
\text{Vol}_g(B_x(1)) &\leq \text{Vol}_g((t-1, t+1) \times N) \\
&\leq \text{Vol}_{d_{S_N^2}}(N) \int_{t-1}^{t+1} f(t) dt \\
&= 2\text{Vol}_{d_{S_N^2}}(N) f(\xi) \\
&\rightarrow 0 \text{ as } t \rightarrow +\infty,
\end{aligned} \tag{3.9}$$

where we used the integral mean value theorem, ξ is some point in $(t-1, t+1)$. This gives the desired result. \square

4. Sufficient conditions

In this section, we investigate sufficient conditions under which Trudinger-Moser inequality holds. Precisely we shall prove theorem 2.3 and proposition 2.4. Firstly we have the following key observation:

Lemma 4.1. *Let $\mathbb{B}_0(\delta) \subset \mathbb{R}^n$ be a ball centered at 0 with radius δ . If $0 \leq \alpha \leq \alpha_n = n\omega_{n-1}^{1/(n-1)}$, then there exists some constant C depending only on n such that for all $u \in W_0^{1,n}(\mathbb{B}_0(\delta))$ satisfying $\int_{\mathbb{B}_0(\delta)} |\nabla u|^n dx \leq 1$, there holds*

$$\int_{\mathbb{B}_0(\delta)} \zeta\left(n, \alpha|u|^{\frac{n}{n-1}}\right) dx \leq C\delta^n \left(\frac{\alpha}{\alpha_n}\right)^{n-1} \int_{\mathbb{B}_0(\delta)} |\nabla u|^n dx. \tag{4.1}$$

Proof. Let $\tilde{u} = u/\|\nabla u\|_{L^n(\mathbb{B}_0(\delta))}$. Since $\|\nabla u\|_{L^n(\mathbb{B}_0(\delta))} \leq 1$ and $0 \leq \alpha \leq \alpha_n$, we have

$$\begin{aligned}
\zeta\left(n, \alpha|u|^{\frac{n}{n-1}}\right) &= \sum_{k=n-1}^{\infty} \frac{\alpha^k |u|^{\frac{nk}{n-1}}}{k!} \\
&= \sum_{k=n-1}^{\infty} \left(\frac{\alpha}{\alpha_n}\right)^k \frac{\alpha_n^k \|\nabla u\|_{L^n(\mathbb{B}_0(\delta))}^{\frac{nk}{n-1}} |\tilde{u}|^{\frac{nk}{n-1}}}{k!} \\
&\leq \|\nabla u\|_{L^n(\mathbb{B}_0(\delta))}^n \left(\frac{\alpha}{\alpha_n}\right)^{n-1} \zeta\left(n, \alpha_n |\tilde{u}|^{\frac{n}{n-1}}\right).
\end{aligned} \tag{4.2}$$

It follows from the classical Trudinger-Moser inequality ((1.2) with α_0 replaced by α_n) that

$$\int_{\mathbb{B}_0(\delta)} \zeta\left(n, \alpha_n |\tilde{u}|^{\frac{n}{n-1}}\right) dx \leq C\delta^n \tag{4.3}$$

for some constant C depending only on n . Integrating (4.2) on $\mathbb{B}_0(\delta)$, we immediately obtain (4.1) by using (4.3). This concludes the lemma. \square

Let (M, g) be a complete Riemannian n -manifold with $\text{Ric}_{(M,g)} \geq Kg$ for some $K \in \mathbb{R}$ and $\text{inj}_{(M,g)} \geq i_0$ for some $i_0 > 0$. Then we have the following local version of Trudinger-moser inequality which is the key estimate for the proof of theorem 2.3:

Lemma 4.2. For any $\alpha : 0 < \alpha < \alpha_n$ there exists some constant δ depending only on n, α, K and i_0 such that for all $x \in M$ and all $u \in C_0^\infty(B_x(\delta))$ with $\|\nabla_g u\|_{L^n(B_x(\delta))} \leq 1$, there holds

$$\int_M \zeta\left(n, \alpha|u|^{\frac{n}{n-1}}\right) dv_g \leq C \int_M |\nabla_g u|^n dv_g$$

for some constant C depending only on n, α, K and i_0 .

Proof. By (Hebey [18], theorem 1.3), we know that for any $\epsilon > 0$ there exists a positive constant δ depending only on ϵ, n, K and i_0 satisfying the following property: for any $x \in M$ there exists a harmonic coordinate chart $\phi : B_x(\delta) \rightarrow \mathbb{R}^n$ such that $\phi(x) = 0$, and the components (g_{jl}) of g in this chart satisfy

$$e^{-\epsilon} \delta_{jl} \leq g_{jl} \leq e^\epsilon \delta_{jl}$$

as bilinear forms. One then has that $\phi(B_x(\delta)) \subset \mathbb{B}_0(e^{\epsilon/2}\delta)$. Let u be a function in $C_0^\infty(B_x(\delta))$ and $\|\nabla_g u\|_{L^n(B_x(\delta))} \leq 1$. It is not difficult to see that

$$\int_{B_x(\delta)} |\nabla_g u|^n dv_g \geq e^{-n\epsilon} \int_{\mathbb{B}_0(e^{\epsilon/2}\delta)} |\nabla(u \circ \phi^{-1})(x)|^n dx, \quad (4.4)$$

$$\int_M \zeta\left(n, \alpha|u|^{\frac{n}{n-1}}\right) dv_g \leq e^{n\epsilon/2} \int_{\mathbb{B}_0(e^{\epsilon/2}\delta)} \zeta\left(n, \alpha|(u \circ \phi^{-1})(x)|^{\frac{n}{n-1}}\right) dx. \quad (4.5)$$

For any fixed $\alpha : 0 < \alpha < \alpha_n$, there exists some ϵ_0 depending only on n and α such that when $0 < \epsilon \leq \epsilon_0$, it follows from (4.4) and $\|\nabla_g u\|_{L^n(B_x(\delta))} \leq 1$ that

$$\alpha \left(\int_{\mathbb{B}_0(e^{\epsilon/2}\delta)} |\nabla(u \circ \phi^{-1})(x)|^n dx \right)^{1/(n-1)} \leq \alpha e^{n\epsilon_0/(n-1)} < \alpha_n.$$

Now let $\epsilon = \epsilon_0$ be fixed and δ depending only on ϵ_0, n, K and i_0 be chosen as above. By lemma 4.1, there exists a constant $C_1 = C_1(n)$ depending only on n such that

$$\int_{\mathbb{B}_0(e^{\epsilon_0/2}\delta)} \zeta\left(n, \alpha|(u \circ \phi^{-1})(x)|^{\frac{n}{n-1}}\right) dx \leq C_1 e^{n\epsilon_0/2} \delta^n \int_{\mathbb{B}_0(e^{\epsilon_0/2}\delta)} |\nabla(u \circ \phi^{-1})(x)|^n dx.$$

This together with (4.4) and (4.5) implies that

$$\int_M \zeta\left(n, \alpha|u|^{\frac{n}{n-1}}\right) dv_g \leq C_1 e^{2n\epsilon_0} \delta^n \int_M |\nabla u|^n dv_g.$$

Take $C = C_1 e^{2n\epsilon_0} \delta^n$. We conclude that C depends on n, α, K and i_0 . \square

Proof of theorem 2.3. (i) For any $\alpha : 0 < \alpha < \alpha_n$, let $\delta = \delta(n, \alpha, K, i_0)$ be chosen as in lemma 4.2. Independently, by Gromov's covering lemma (Hebey [18], lemma 1.6), we can select a sequence (x_j) of points of M such that

- (a) $M = \cup_j B_{x_j}(\delta/2)$, and for any $j \neq l$ there holds $B_{x_j}(\delta/4) \cap B_{x_l}(\delta/4) = \emptyset$;
- (b) there exists N depending only on n, K and δ such that each point of M has a neighborhood which intersects at most N of the $B_{x_j}(\delta)$'s.

For any j , we take a cut-off function $\phi_j \in C_0^\infty(B_{x_j}(\delta))$ satisfying $0 \leq \phi_j \leq 1$, $\phi_j \equiv 1$ on $B_{x_j}(\delta/2)$, and $|\nabla_g \phi_j| \leq 4/\delta$. It follows that for all j

$$|\nabla_g \phi_j^2| = 2\phi_j |\nabla_g \phi_j| \leq \frac{8}{\delta} \phi_j. \quad (4.6)$$

By the covering properties (a) and (b), we have

$$1 \leq \sum_j \phi_j(x) \leq N \text{ for all } x \in M. \quad (4.7)$$

Set $\tau = 8/\delta$. Assume $u \in C_0^\infty(M)$ satisfies

$$\|u\|_{1,\tau} = \left(\int_M |\nabla u|^n dv_g \right)^{1/n} + \tau \left(\int_M |u|^n dv_g \right)^{1/n} \leq 1.$$

It follows from (4.6) and the Minkowsky inequality that

$$\left(\int_M |\nabla_g(\phi_j^2 u)|^n dv_g \right)^{1/n} \leq \left(\int_M \phi_j^{2n} |\nabla_g u|^n dv_g \right)^{1/n} + \left(\int_M |\nabla_g \phi_j^2|^n |u|^n dv_g \right)^{1/n} \leq \|u\|_{1,\tau} \leq 1.$$

In view of lemma 4.2, this leads to

$$\begin{aligned} \int_M \zeta\left(n, \alpha |u|^{\frac{n}{n-1}}\right) dv_g &\leq \sum_j \int_{B_{\delta/2}(x_j)} \zeta\left(n, \alpha |u|^{\frac{n}{n-1}}\right) dv_g \\ &\leq \sum_j \int_{B_\delta(x_j)} \zeta\left(n, \alpha |\phi_j^2 u|^{\frac{n}{n-1}}\right) dv_g \\ &\leq C \sum_j \int_M |\nabla(\phi_j^2 u)|^n dv_g \end{aligned} \quad (4.8)$$

for some constant C depending only on n, α, K and i_0 . In addition we have by using (4.6) and $0 \leq \phi_j \leq 1$ that

$$\begin{aligned} \int_M |\nabla_g(\phi_j^2 u)|^n dv_g &\leq 2^n \int_M (\phi_j^{2n} |\nabla_g u|^n + |\nabla_g \phi_j^2|^n |u|^n) dv_g \\ &\leq 2^n \int_M \phi_j |\nabla_g u|^n dv_g + \frac{16^n}{\delta^n} \int_M \phi_j |u|^n dv_g. \end{aligned}$$

In view of (4.7), it follows that

$$\begin{aligned} \sum_j \int_M |\nabla_g(\phi_j^2 u)|^n dv_g &\leq 2^n N \int_M |\nabla_g u|^n dv_g + \frac{16^n}{\delta^n} N \int_M |u|^n dv_g \\ &\leq 2^n N + \frac{16^n}{\tau \delta^n} N. \end{aligned}$$

This together with (4.8) implies

$$\int_M \zeta\left(n, \alpha |u|^{\frac{n}{n-1}}\right) dv_g \leq C$$

for some constant C depending only on n, α, K and i_0 . By the density of $C_0^\infty(M)$ in $W^{1,n}(M)$, the inequality (1.7) holds for the above α, τ and C .

By proposition 2.1, we have that $W^{1,n}(M)$ is continuously embedded in $L^q(M)$ for any $q \geq n$.

(ii) Fix some point $z \in M$, let $r = r(x) = d_g(z, x)$ be the geodesic distance between x and z . Without loss of generality, we may assume the injectivity radius of (M, g) at z is strictly larger than 1. Take a function sequence

$$\phi_\epsilon(x) = \begin{cases} 1, & \text{when } r < \epsilon \\ \left(\log \frac{1}{\epsilon}\right)^{-1} \log \frac{1}{r}, & \text{when } \epsilon \leq r \leq 1 \\ 0, & \text{when } r > 1. \end{cases}$$

Then $\phi_\epsilon \in W^{1,n}(M)$ and for any constant $\tau > 0$ there holds

$$\|\phi_\epsilon\|_{1,\tau} = \left(\log \frac{1}{\epsilon}\right)^{(1-n)/n} \omega_{n-1}^{1/n} \left(1 + O\left(\frac{1}{\log \epsilon}\right)\right).$$

Set $\tilde{\phi}_\epsilon = \phi_\epsilon / \|\phi_\epsilon\|_{1,\tau}$. Then we have on the geodesic ball $B_z(\epsilon) \subset M$,

$$\zeta(n, \alpha \tilde{\phi}_\epsilon^{\frac{n}{n-1}}) = e^{\alpha \tilde{\phi}_\epsilon^{\frac{n}{n-1}}} - \sum_{k=0}^{n-2} \frac{\alpha^k \tilde{\phi}_\epsilon^{\frac{nk}{n-1}}}{k!} \geq \epsilon^{\alpha \omega_{n-1}^{-\frac{1}{n-1}} (1+O(1/\log \epsilon))} + O\left(\left(\log \frac{1}{\epsilon}\right)^{n-2}\right).$$

Note that $\alpha \omega_{n-1}^{-\frac{1}{n-1}} > n$ for any $\alpha > \alpha_n$. Hence, when $\alpha > \alpha_n$, we have

$$\begin{aligned} \int_M \zeta(n, \alpha |\tilde{\phi}_\epsilon|^{\frac{n}{n-1}}) dv_g &\geq \int_{B_z(\epsilon)} \zeta(n, \alpha |\tilde{\phi}_\epsilon|^{\frac{n}{n-1}}) dv_g \\ &\geq \frac{\omega_{n-1}}{n} (1 + o_\epsilon(1)) \epsilon^{n - \alpha \omega_{n-1}^{-1/(n-1)} (1+O(1/\log \epsilon))} + o_\epsilon(1). \\ &\rightarrow +\infty \quad \text{as } \epsilon \rightarrow 0. \end{aligned}$$

This ends the proof of (ii).

(iii) Take $\alpha_0 : 0 < \alpha_0 < \alpha_n$. By (i) there exists some $\tau_0 = \tau_0(n, \alpha_0, K, i_0) > 0$ such that

$$\Lambda_{\alpha_0} := \sup_{\|u\|_{1,\tau_0} \leq 1} \int_M \zeta(n, \alpha_0 |u|^{\frac{n}{n-1}}) dv_g < \infty.$$

Given any $\alpha > 0$ and any $u \in W^{1,n}(M)$. Since $C_0^\infty(M)$ is dense in $W^{1,n}(M)$ under the norm $\|\cdot\|_{W^{1,n}(M)}$, which is equivalent to the norm $\|\cdot\|_{1,\tau_0}$, we can choose some $u_0 \in C_0^\infty(M)$ such that

$$2^{\frac{n}{n-1}} \alpha \|u - u_0\|_{1,\tau_0}^{\frac{n}{n-1}} < \alpha_0. \quad (4.9)$$

Since $\zeta(n, t)$ is increasing in t for $t \geq 0$, we obtain by using (2.3)

$$\begin{aligned} \int_M \zeta(n, \alpha |u|^{\frac{n}{n-1}}) dv_g &\leq \int_M \zeta(n, 2^{\frac{n}{n-1}} \alpha |u - u_0|^{\frac{n}{n-1}} + 2^{\frac{n}{n-1}} \alpha |u_0|^{\frac{n}{n-1}}) dv_g \\ &\leq \frac{1}{\mu} \int_M \zeta(n, 2^{\frac{n}{n-1}} \alpha \mu |u - u_0|^{\frac{n}{n-1}}) dv_g \\ &\quad + \frac{1}{\nu} \int_M \zeta(n, 2^{\frac{n}{n-1}} \alpha \nu |u_0|^{\frac{n}{n-1}}) dv_g, \end{aligned} \quad (4.10)$$

where $1/\mu + 1/\nu = 1$. In view of (4.9), we can take $\mu > 1$ sufficiently close to 1 such that

$$2^{\frac{n}{n-1}} \alpha \mu \|u - u_0\|_{1, \tau_0}^{\frac{n}{n-1}} < \alpha_0.$$

Hence

$$\int_M \zeta(n, 2^{\frac{n}{n-1}} \alpha \mu |u - u_0|^{\frac{n}{n-1}}) dv_g \leq \Lambda_{\alpha_0}. \quad (4.11)$$

Since $u_0 \in C_0^\infty(M)$, particularly u_0 has compact support, there holds

$$\int_M \zeta(n, 2^{\frac{n}{n-1}} \alpha \nu |u_0|^{\frac{n}{n-1}}) dv_g < \infty. \quad (4.12)$$

Combining (4.10), (4.11) and (4.12), we obtain

$$\int_M \zeta(n, \alpha |u|^{\frac{n}{n-1}}) dv_g < \infty.$$

This completes the proof of (iii). \square

Now we shall prove proposition 2.4. Let us recall some notations from Riemannian geometry. In any chart, the Christoffel symbols of the Levi-Civita connection are given by

$$\Gamma_{ij}^k = \frac{1}{2} g^{mk} (\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ij}), \quad (4.13)$$

where g_{ij} 's are the components of g , (g^{ij}) denotes the inverse matrix of (g_{ij}) . Here and in the sequel the Einstein's summation convention is adopted. Denote the Riemannian curvature of (M, g) , a $(4, 0)$ -type tensor field, by $\text{Rm}_{(M, g)}$. The components of $\text{Rm}_{(M, g)}$ are given by the relation

$$R_{ijkl} = g_{i\alpha} (\partial_k \Gamma_{jl}^\alpha - \partial_l \Gamma_{jk}^\alpha + \Gamma_{k\beta}^\alpha \Gamma_{jl}^\beta - \Gamma_{l\beta}^\alpha \Gamma_{jk}^\beta). \quad (4.14)$$

Similarly, the components of the Ricci curvature $\text{Rc}_{(M, g)}$ of (M, g) are given by the relation

$$R_{ij} = g^{\alpha\beta} R_{i\alpha j\beta}. \quad (4.15)$$

Proof of proposition 2.4. In view of proposition 2.1, it suffices to construct a complete non-compact Riemannian n -manifold (M, g) such that its Ricci curvature has lower bound and there holds

$$\inf_{x \in M} \text{Vol}_g(B_x(1)) = 0.$$

Again we consider the warped product

$$M = \mathbb{R} \times N, \quad g(x, \theta) = dx^2 + f(x) ds_N^2,$$

where (N, ds_N^2) is a compact $(n-1)$ -Riemannian manifold, dx^2 is the euclidean metric of \mathbb{R} , and f is a smooth function satisfying $f(x) > 0, \forall x \in \mathbb{R}$. In the following we calculate the Ricci curvature of (M, g) . In some product chart $(\mathbb{R} \times U, Id \times \phi) (\{x, y^2, \dots, y^n\})$, $g_{11} = 1, g_{1\alpha} = 0, g_{\alpha\beta} = f h_{\alpha\beta}, g^{11} = 1, g^{1,\alpha} = 0$, and $g^{\alpha\beta} = f^{-1} h^{\alpha\beta}$. Equivalently

$$g = dx^2 + f(x) h_{\alpha\beta} dy^\alpha dy^\beta,$$

where $(h_{\alpha\beta})$ denote components of the metric ds_N^2 . Here and in the sequel, all indices α, β, μ, ν and λ run from 2 to n . In view of (4.13), the Christoffel symbols of the Levi-Civita connection was calculated as follows:

$$\begin{aligned}\Gamma_{11}^1 &= \Gamma_{11}^\alpha = \Gamma_{1\alpha}^1 = 0, \quad \Gamma_{1\alpha}^\beta = \frac{1}{2}g^{\mu\beta}\partial_1 g_{\mu\alpha} = \frac{f'}{2f}\delta_\alpha^\beta \\ \Gamma_{\alpha\beta}^1 &= -\frac{1}{2}\partial_1 g_{\alpha\beta} = -\frac{f'}{2}h_{\alpha\beta}, \quad \Gamma_{\alpha\beta}^\gamma = \widetilde{\Gamma}_{\alpha\beta}^\gamma,\end{aligned}$$

where δ_α^β is equal to 1 when $\alpha = \beta$, and 0 when $\alpha \neq \beta$, $\widetilde{\Gamma}_{\alpha\beta}^\gamma$'s are components of the Christoffel symbols of Levi-Civita connection on (N, ds_N^2) . In view of (4.14), the components of the Riemannian curvature reads

$$\begin{aligned}R_{1\alpha 1\beta} &= g_{11}\partial_1 \Gamma_{\alpha\beta}^1 \\ &= \frac{f'^2 - 2ff''}{4f}h_{\alpha\beta} \\ R_{1\alpha\beta\gamma} &= g_{11}(\partial_\beta \Gamma_{\alpha\gamma}^1 - \partial_\gamma \Gamma_{\alpha\beta}^1 + \Gamma_{\beta k}^1 \Gamma_{\alpha\gamma}^k - \Gamma_{\gamma k}^1 \Gamma_{\alpha\beta}^k) \\ &= \frac{f'}{2}(-\partial_\beta h_{\alpha\gamma} + \partial_\gamma h_{\alpha\beta} - h_{\beta\mu}\widetilde{\Gamma}_{\alpha\gamma}^\mu + h_{\gamma\mu}\widetilde{\Gamma}_{\alpha\beta}^\mu) \\ R_{\alpha\beta\gamma\mu} &= g_{\alpha\lambda}(\partial_\gamma \Gamma_{\beta\mu}^\lambda - \partial_\mu \Gamma_{\beta\gamma}^\lambda + \Gamma_{\gamma k}^\lambda \Gamma_{\beta\mu}^k - \Gamma_{\mu k}^\lambda \Gamma_{\beta\gamma}^k) \\ &= f\widetilde{R}_{\alpha\beta\gamma\mu} + g_{\alpha\lambda}(\Gamma_{\gamma 1}^\lambda \Gamma_{\beta\mu}^1 - \Gamma_{\mu 1}^\lambda \Gamma_{\beta\gamma}^1) \\ &= f\widetilde{R}_{\alpha\beta\gamma\mu} + \frac{f'^2}{4}(h_{\alpha\mu}h_{\beta\gamma} - h_{\alpha\gamma}h_{\beta\mu}),\end{aligned}$$

where $\widetilde{R}_{\alpha\beta\gamma\mu}$'s denote the components of Riemannian curvature of (N, ds_N^2) . In view of (4.15), we get the components of the Ricci curvature as follows.

$$\begin{aligned}R_{11} &= g^{\alpha\beta}R_{1\alpha 1\beta} \\ &= (n-1)\frac{f'^2 - 2ff''}{4f^2} \\ R_{1\alpha} &= g^{\beta\gamma}R_{1\beta\alpha\gamma} \\ &= \frac{f'}{2f}h^{\beta\gamma}(-\partial_\alpha h_{\beta\gamma} + \partial_\gamma h_{\alpha\beta} - h_{\alpha\mu}\widetilde{\Gamma}_{\beta\gamma}^\mu + h_{\gamma\mu}\widetilde{\Gamma}_{\alpha\beta}^\mu) \\ R_{\alpha\beta} &= g^{11}R_{\alpha 1\beta 1} + g^{\mu\nu}R_{\alpha\mu\beta\nu} \\ &= \frac{f'^2 - 2ff''}{4f}h_{\alpha\beta} + \widetilde{R}_{\alpha\beta} + \frac{f'^2}{4f}h^{\mu\nu}(h_{\alpha\nu}h_{\mu\beta} - h_{\alpha\beta}h_{\mu\nu}) \\ &= \frac{(2-n)f'^2 - 2ff''}{4f}h_{\alpha\beta} + \widetilde{R}_{\alpha\beta},\end{aligned}$$

where $\widetilde{R}_{\alpha\beta}$'s are components of the Ricci curvature of (N, ds_N^2) . If we assume the functions f , f'/f and f''/f are all bounded, then in the chart $(\mathbb{R} \times U, Id \times \phi)$, the eigenvalues of the matrix (R_{jl}) and the matrix (g_{jl}) are uniformly bounded. Thus there exists some constant $K_1 \in \mathbb{R}$ such that $(R_{jl}) \geq K_1(g_{jl})$. Note that (N, ds_N^2) is compact. There exists some constant $K \in \mathbb{R}$ such that $Ric_{(M,g)} \geq Kg$ as bilinear forms. If we further assume $\lim_{x \rightarrow +\infty} f(x) = 0$, then by (3.9), we have

$\text{Vol}_g(B_y(1)) \rightarrow 0$ as $x \rightarrow +\infty$, where $y = (x, m) \in \mathbb{R} \times N$. One can check that the following functions satisfy all the above assumptions on f .

- f is a smooth positive function defined on \mathbb{R} and satisfies

$$f(x) = \begin{cases} (1+x^2)e^{-x+\sin x} & \text{when } x > 1 \\ 1, & \text{when } x < 0 \end{cases}$$

- f is a smooth positive function defined on \mathbb{R} and satisfies

$$f(x) = \begin{cases} \frac{1}{\log x} & \text{when } x > 2 \\ 1, & \text{when } x < 0. \end{cases}$$

This gives the desired result. \square

5. Proof of proposition 2.5

In this section, we shall construct complete noncompact Riemannian n -manifolds to show that the condition Ricci curvature has lower bound in theorem 2.3 is not necessarily needed.

Proof of proposition 2.5. It suffices to construct a complete noncompact Riemannian n -manifold on which Trudinger-Moser embedding holds, but its Ricci curvature has no lower bound. For this purpose, we consider the Riemannian manifold (\mathbb{R}^n, g) , where \mathbb{R}^n is the euclidian space and

$$g = dx_1^2 + f(x_1)dx_2^2 + \cdots + f(x_1)dx_n^2,$$

and f is a smooth function on \mathbb{R} such that $a \leq f \leq b$ for two positive constants a and b . Clearly (\mathbb{R}^n, g) is complete and noncompact. In view of Trudinger-Moser inequality on the standard euclidian space \mathbb{R}^n [8, 12, 32], one can easily see that if α is chosen sufficiently small, then the supremum

$$\sup_{u \in W^{1,n}(M), \|u\|_{W^{1,n}} \leq 1} \int_{\mathbb{R}^n} \zeta\left(n, \alpha |u|^{\frac{n}{n-1}}\right) dv_g$$

is finite, i.e. Trudinger-Moser inequality holds on the manifold (\mathbb{R}^n, g) , where

$$\|u\|_{W^{1,n}} = \left(\int_{\mathbb{R}^n} (|\nabla_g u|^n + |u|^n) dv_g \right)^{\frac{1}{n}}.$$

In the following, we shall further choose f such that the Ricci curvature of (\mathbb{R}^n, g) is unbounded from below. By (4.15),

$$R_{11} = (n-1) \frac{f'^2 - 2ff''}{4f^2}. \quad (5.1)$$

It suffices to find a sequence of points $(x^{(m)})$ of \mathbb{R}^n such that $R_{11}(x^{(m)}) \rightarrow -\infty$. One choice of f is that $f(t) = 2 + \sin t^2$. In this case, we have

$$f'(x_1) = 2 + 2x_1 \cos x_1^2, \quad f''(x_1) = 2 \cos x_1^2 - 4x_1^2 \sin x_1^2.$$

Thus (5.1) implies

$$R_{11}(x) = (n-1) \frac{(2 + 2x_1 \cos x_1^2)^2 - 2(2 + \sin x_1^2)(2 \cos x_1^2 - 4x_1^2 \sin x_1^2)}{4(2 + \sin x_1^2)^2}.$$

Choosing $x^{(m)} = (\sqrt{2m\pi + 3\pi/2}, 0, \dots, 0)$, we obtain

$$R_{11}(x^{(m)}) = -4m\pi - 3\pi + n - 1 \rightarrow -\infty \quad \text{as } m \rightarrow \infty.$$

Another choice of f is that $f(t) = e^{\sin t^2}$. In this case, we have

$$f'(x_1) = 2x_1 e^{\sin x_1^2} \cos x_1^2, \quad f''(x_1) = e^{\sin x_1^2} (-4x_1^2 \sin x_1^2 + 4x_1^2 \cos^2 x_1^2 + 2 \cos x_1^2).$$

In view of (5.1), we obtain

$$R_{11}(x) = (n-1)(2x_1^2 \sin x_1^2 + x_1^2 \cos^2 x_1^2 - 2x_1^2 \cos^2 x_1^2 - \cos x_1^2).$$

Again, we select $x^{(m)} = (\sqrt{2m\pi + 3\pi/2}, 0, \dots, 0)$ and conclude $R_{11}(x^{(m)}) \rightarrow -\infty$ as $m \rightarrow \infty$. \square

6. Adams inequalities

In this section, we concern Adams inequalities on complete noncompact Riemannian manifolds. Precisely we shall prove theorem 2.6. The method we adopted here is similar to that of theorem 2.3.

Proof of theorem 2.6. (i) Suppose that $\text{inj}_{(M,g)} \geq i_0 > 0$ and there exist constants $C(k)$ such that $|\nabla^k \text{Rc}_{(M,g)}| \leq C(k)$, $k = 0, 1, \dots, m-1$. It follows from (Hebey [18], theorem 1.3) that for any $Q > 1$ and $\alpha \in (0, 1)$, the harmonic radius $r_H = r_H(Q, m, \alpha)$ is positive. Namely, for any $Q > 1$, $\alpha \in (0, 1)$, and $x \in M$, there exists a harmonic coordinate chart $\psi : B_x(r_H) \rightarrow \mathbb{R}^n$ such that

$$\begin{cases} Q^{-1} \delta_{lq} \leq g_{lq} \leq Q \delta_{lq} & \text{as a bilinear form;} \\ \sum_{1 \leq |\beta| \leq m} \|\partial^\beta g_{lq}\|_{C^0(B_x(r_H))} + \sum_{|\beta|=m} \|\partial^\beta g_{lq}\|_{C^0(B_x(r_H))} \leq Q - 1. \end{cases} \quad (6.1)$$

Now we fix $Q > 1$ and $\alpha \in (0, 1)$. Without loss of generality, we may assume $\psi(x) = 0$. Particularly we have that for any $r : 0 < r \leq r_H$

$$\mathbb{B}_0(r/\sqrt{Q}) \subset \psi(B_x(r)) \subset \mathbb{B}_0(\sqrt{Q}r).$$

Let $\eta \in C_0^\infty(\mathbb{R}^n)$ be such that $0 \leq \eta \leq 1$, and

$$\eta = \begin{cases} 1 & \text{on } \mathbb{B}_0(r_H/(4\sqrt{Q})), \\ 0 & \text{on } \mathbb{R}^n \setminus \mathbb{B}_0(r_H/(2\sqrt{Q})). \end{cases}$$

Then $\eta \circ \psi \in C_0^\infty(M)$ satisfies $0 \leq \eta \circ \psi \leq 1$, $\eta \circ \psi \equiv 1$ on $B_x(r_H/(4Q))$, and $\eta \circ \psi \equiv 0$ on $M \setminus B_x(r_H/2)$. By Gromov's covering lemma (Hebey [18], lemma 1.6), there exists a sequence of points (x_k) of M such that

$$M = \cup_k B_{x_k}(r_H/(4Q)) \quad (6.2)$$

and there exists some integer N such that for any $x \in M$, x belongs to at most N balls in the covering. Let $\psi_k : B_{x_k}(r_H) \rightarrow \mathbb{R}^n$ be as the above ψ and set $\eta_k = \eta \circ \psi_k$. By (6.1), the components of the metric tensor are C^m -controlled in the charts $(B_{x_k}(r_H), \psi_k)$. It then follows that there exists some constant $C_1 > 0$ depending only on r_H and Q such that $|\nabla_g^l \eta_k| \leq C_1$ for all $l : 0 \leq l \leq m$ and all $k \in \mathbb{N}$, where ∇_g^l is defined by (1.3).

Assume $u \in C^\infty(M)$ satisfies $\|u\|_{W^{m,n/m}(M)} \leq 1$. Then we get

$$\eta_k^{m+1} u \in C_0^\infty(B_{x_k}(r_H/2))$$

and

$$\|\nabla_g^m(\eta_k^{m+1} u)\|_{L^{\frac{n}{m}}(B_{x_k}(r_H/2))} \leq C_2 \quad (6.3)$$

for some constant C_2 depending only on n, m , and C_1 . By the standard elliptic estimates (Gilbarg-Trudinger [16], Chapter 9), one can see that

$$\|\nabla_{\mathbb{R}^n}^m((\eta_k^{m+1} u) \circ \psi_k^{-1})\|_{L^{\frac{n}{m}}(\mathbb{B}_0(\sqrt{Q}r_H))} \leq C_3 \quad (6.4)$$

for some constant C_3 depending only on n, m, Q, r_H and C_1 . Let j be the smallest integer great than or equal to n/m . Similarly as we derived (4.8), we calculate by using (6.2), (6.3) and the relation $(j-1)n/(n-m) \geq n/m$

$$\begin{aligned} \int_M \zeta\left(j, \alpha|u|^{\frac{n}{n-m}}\right) dv_g &\leq \sum_k \int_{B_{x_k}(r_H/(4Q))} \zeta\left(j, \alpha|u|^{\frac{n}{n-m}}\right) dv_g \\ &\leq \sum_k \int_{B_{x_k}(r_H/2)} \zeta\left(j, \alpha|\eta_k^{m+1} u|^{\frac{n}{n-m}}\right) dv_g \\ &\leq \sum_k \left(\frac{\|\nabla_g^m(\eta_k^{m+1} u)\|_{L^{\frac{n}{m}}(B_{x_k}(r_H/2))}}{C_2} \right)^{\frac{(j-1)n}{n-m}} \int_{B_{x_k}(r_H/2)} \zeta\left(j, \alpha C_2^{\frac{n}{n-m}} |\eta_k^{m+1} u|^{\frac{n}{n-m}}\right) dv_g \\ &\leq \sum_k \frac{\|\nabla_g^m(\eta_k^{m+1} u)\|_{L^{\frac{n}{m}}(B_{x_k}(r_H/2))}^{\frac{n}{m}}}{C_2^{\frac{n}{m}}} \int_{B_{x_k}(r_H/2)} \zeta\left(j, \alpha C_2^{\frac{n}{n-m}} |\eta_k^{m+1} u|^{\frac{n}{n-m}}\right) dv_g. \end{aligned} \quad (6.5)$$

Noting that $Q^{-1}\delta_{lq} \leq g_{lq} \leq Q\delta_{lq}$ as a bilinear form, we have

$$\int_{B_{x_k}(r_H/2)} \zeta\left(j, \alpha C_2^{\frac{n}{n-m}} |\eta_k^{m+1} u|^{\frac{n}{n-m}}\right) dv_g \leq Q^{\frac{n}{2}} \int_{\mathbb{B}_0(\sqrt{Q}r_H)} \zeta\left(j, \alpha C_2^{\frac{n}{n-m}} |(\eta_k^{m+1} u) \circ \psi_k^{-1}|^{\frac{n}{n-m}}\right) dx. \quad (6.6)$$

In view of (6.4), we take

$$\alpha_0 = \beta_0 / (C_2 C_3)^{\frac{n}{n-m}}. \quad (6.7)$$

Then for any $\alpha : 0 < \alpha \leq \alpha_0$, it follows from Adams inequality (1.4) that

$$\int_{\mathbb{B}_0(\sqrt{Q}r_H)} \zeta\left(j, \alpha C_2^{\frac{n}{n-m}} |(\eta_k^{m+1} u) \circ \psi_k^{-1}|^{\frac{n}{n-m}}\right) dx \leq C_{m,n} |\mathbb{B}_0(\sqrt{Q}r_H)|. \quad (6.8)$$

Clearly there exists some constant $C_4 > 0$ depending only on n, m, Q and r_H such that

$$|\nabla_g^l \eta_k^{m+1}|^{\frac{n}{m}} \leq C_4 \eta_k, \quad \forall l = 0, 1, \dots, m. \quad (6.9)$$

Since $1 \leq \sum_k \eta_k(x) \leq N$ for all $x \in M$, we obtain by combining (6.5)-(6.9) that

$$\begin{aligned}
\int_M \zeta\left(j, \alpha|u|^{\frac{n}{n-m}}\right) dv_g &\leq C_5 \sum_k \int_M |\nabla_g^m(\eta_k^{m+1}u)|^{\frac{n}{m}} dv_g \\
&\leq C_5 \sum_k \sum_{l=0}^m (C_m^l)^{\frac{n}{m}} \int_M |\nabla_g^{m-k} \eta_k^{m+1} \nabla_g^l u|^{\frac{n}{m}} dv_g \\
&\leq C_4 C_5 \sum_{l=0}^m (C_m^l)^{\frac{n}{m}} \int_M (\sum_k \eta_k) |\nabla_g^l u|^{\frac{n}{m}} dv_g \\
&\leq C_4 C_5 N \sum_{l=0}^m (C_m^l)^{\frac{n}{m}} \int_M |\nabla_g^l u|^{\frac{n}{m}} dv_g \\
&\leq C_6
\end{aligned}$$

for constants C_5 and C_6 depending only on n, m, Q and r_H , where $C_m^l = \frac{m!}{l!(m-l)!}$.

According to (Hebey [18], theorem 2.8), $C_0^\infty(M)$ is dense in $W^{m, \frac{n}{m}}(M)$. Hence for any $u \in W^{m, \frac{n}{m}}(M)$, there exists a sequence (u_k) in $C_0^\infty(M)$ such that $\|u_k - u\|_{W^{m, \frac{n}{m}}(M)} \rightarrow 0$ as $k \rightarrow \infty$. Assume $\|u\|_{W^{m, \frac{n}{m}}(M)} \leq 1$. Then for any $\alpha : 0 < \alpha < \alpha_0$ there holds

$$\int_M \zeta\left(j, \alpha|u|^{\frac{n}{n-m}}\right) dv_g \leq \lim_{k \rightarrow \infty} \int_M \zeta\left(j, \alpha|u_k|^{\frac{n}{n-m}}\right) dv_g \leq C_6.$$

Using the same method of deriving $W^{1,n}(M) \hookrightarrow L^q(M)$ continuously for all $q \geq n$ in theorem 2.3, we obtain the continuous embedding $W^{m,n/m}(M) \hookrightarrow L^q(M)$ for any $q \geq n/m$.

(ii) Let $\alpha > 0$ be any real number and u be any function belonging to the space $W^{m, \frac{n}{m}}(M)$. Since $C_0^\infty(M)$ is dense in $W^{m, \frac{n}{m}}(M)$, there exists some $u_0 \in C_0^\infty(M)$ such that

$$\alpha \|u - u_0\|_{W^{m, \frac{n}{m}}(M)}^{\frac{n}{n-m}} < \alpha_0/2, \quad (6.10)$$

where α_0 is defined by (6.7). Using (2.3) and an elementary inequality

$$|a|^p \leq (1 + \epsilon)|a - b|^p + c(\epsilon, p)|b|^p,$$

where $\epsilon > 0, p > 1$ and $c(\epsilon, p)$ is a constant depending only on ϵ and p , we have

$$\begin{aligned}
\int_M \zeta\left(j, \alpha|u|^{\frac{n}{n-m}}\right) dv_g &\leq \int_M \zeta\left(j, (1 + \epsilon)\alpha|u - u_0|^{\frac{n}{n-m}} + c(\epsilon, n/(n-m))\alpha|u_0|^{\frac{n}{n-m}}\right) dv_g \\
&\leq \frac{1}{\mu} \int_M \zeta\left(j, \mu(1 + \epsilon)\alpha|u - u_0|^{\frac{n}{n-m}}\right) dv_g \\
&\quad + \frac{1}{\nu} \int_M \zeta\left(j, \nu c(\epsilon, n/(n-m))\alpha|u_0|^{\frac{n}{n-m}}\right) dv_g, \quad (6.11)
\end{aligned}$$

where $\mu > 1, \nu > 1$ and $1/\mu + 1/\nu = 1$. Choosing ϵ sufficiently small and μ sufficiently close to 1 such that $\mu(1 + \epsilon)\alpha_0/2 \leq \alpha_0$, in view of (6.10), we have by part (i)

$$\int_M \zeta\left(j, \mu(1 + \epsilon)\alpha|u - u_0|^{\frac{n}{n-m}}\right) dv_g \leq C_6. \quad (6.12)$$

Note that $u_0 \in C_0^\infty(M)$, particularly u_0 has compact support. It follows that

$$\int_M \zeta\left(j, \nu c(\epsilon, n/(n-m))\alpha|u_0|^{\frac{n}{n-m}}\right) dv_g < \infty. \quad (6.13)$$

Inserting (6.12) and (6.13) into (6.11), we complete the proof of part (ii). \square

7. Applications of Trudinger-Moser inequalities

In this section, we consider applications of theorem 2.3, namely the existence and multiplicity results for the problem (2.4) and its perturbation (2.8). Specifically we shall prove theorem 2.7 and theorem 2.10. Throughout this section, we use the notations introduced in section 2. Let (M, g) be a complete noncompact Riemannian n -manifold with $\text{Rc}_{(M,g)} \geq Kg$ for some $K \in \mathbb{R}$ and $\text{inj}_{(M,g)} \geq i_0 > 0$. Assume $\phi(x)$ satisfies the hypotheses (ϕ_1) and (ϕ_2) , $v(x)$ satisfies the hypotheses (v_1) and (v_2) . Let E be a function space defined by (2.5). If $u \in E$, then the E -norm of u is defined by

$$\|u\|_E = \left(\int_M (|\nabla_g u|^n + v|u|^n) dv_g \right)^{1/n}.$$

The following compact embedding result is very important in our analysis.

Proposition 7.1. *For any $q \geq n$, the function space E is compactly embedded in $L^q(M)$.*

Proof. Let (u_k) be a sequence of functions with $\|u_k\|_E \leq C$ for some constant C . It suffices to prove that up to a subsequence, (u_k) converges in $L^q(M)$ for any $q \geq n$. Clearly (u_k) is bounded in $W^{1,n}(M)$, and thus we can assume that for any $q > 1$, up to a subsequence

$$\begin{aligned} u_k &\rightharpoonup u_0 \quad \text{weakly in } E \\ u_k &\rightarrow u_0 \quad \text{strongly in } L_{\text{loc}}^q(M) \\ u_k &\rightarrow u_0 \quad \text{a.e. in } M. \end{aligned} \quad (7.1)$$

If $v(x) \in L^{1/(n-1)}(M)$, using the same argument of ([44], Lemma 2.4), we conclude that $E \hookrightarrow L^q(M)$ compactly for any $q > 1$. So, in view of (v_2) , we may assume $v(x) \rightarrow \infty$ as $d_g(O, x) \rightarrow \infty$, where O is a fixed point of M . Given any $\epsilon > 0$, there exists some $R > 0$ such that $v(x) > (2C)^n/\epsilon$ when $d_g(O, x) \geq R$. Hence

$$\frac{(2C)^n}{\epsilon} \int_{M \setminus B_O(R)} |u_k - u_0|^n dv_g < \int_M v |u_k - u_0|^n dv_g \leq (2C)^n.$$

This gives

$$\int_{M \setminus B_O(R)} |u_k - u_0|^n dv_g < \epsilon.$$

By (7.1), we have

$$\lim_{k \rightarrow \infty} \int_{B_O(R)} |u_k - u_0|^n dv_g = 0.$$

Hence for the above ϵ , there exists some $l \in \mathbb{N}$ such that when $k > l$,

$$\int_M |u_k - u_0|^n dv_g < 2\epsilon.$$

This implies $u_k \rightarrow u_0$ strongly in $L^n(M)$ as $k \rightarrow \infty$.

It follows from (i) of theorem 2.3 that (u_k) is bounded in $L^q(M)$ for any $q \geq n$. Now fixing $q > n$, we get by Hölder's inequality

$$\int_M |u_k - u_0|^q dv_g \leq \left(\int_M |u_k - u_0|^n dv_g \right)^{1/n} \left(\int_M |u_k - u_0|^{\frac{n(q-1)}{n-1}} dv_g \right)^{1-1/n}.$$

This together with the fact that $u_k \rightarrow u_0$ in $L^n(M)$ implies $u_k \rightarrow u_0$ in $L^q(M)$. \square

Let S_q be defined by (2.7). Then we have the following:

Proposition 7.2. *For any $q > n$, S_q is attained by some nonnegative function $u \in E \setminus \{0\}$.*

Proof. Assume $q > n$. It is easy to see that

$$S_q^n = \inf_{\int_M \phi |u|^q dv_g = 1} \int_M (|\nabla u|^n + v|u|^n) dv_g.$$

Choosing a sequence of functions $(u_k) \subset E$ such that $\int_M \phi |u_k|^q dv_g = 1$ and

$$\lim_{k \rightarrow \infty} \int_M (|\nabla u_k|^n + v|u_k|^n) dv_g = S_q^n.$$

By proposition 7.1, there exists some $u \in E$ such that up to a subsequence, $u_k \rightharpoonup u$ weakly in E , $u_k \rightarrow u$ strongly in $L^q(M)$ for any $q \geq n$, and $u_k \rightarrow u$ almost everywhere in M . Since $u_k \rightarrow u$ strongly in $L^s(B_O(R_0))$ for all $s > 1$ and $\phi \in L^p(B_O(R_0))$, we have by using Hölder's inequality that

$$\lim_{k \rightarrow \infty} \int_{B_O(R_0)} \phi |u_k|^q dv_g = \int_{B_O(R_0)} \phi |u|^q dv_g. \quad (7.2)$$

In view of (v_2) , we have

$$\begin{aligned} \int_{M \setminus B_O(R_0)} \phi ||u_k|^q - |u|^q| dv_g &\leq qC_0 \int_M (|u_k|^{q-1} + |u|^{q-1}) |u_k - u| dv_g \\ &\leq qC_0 \left\{ \left(\int_M |u_k|^q dv_g \right)^{1-1/q} + \left(\int_M |u|^q dv_g \right)^{1-1/q} \right\} \\ &\quad \times \left(\int_M |u_k - u|^q dv_g \right)^{1/q} \\ &\rightarrow 0 \text{ as } k \rightarrow \infty. \end{aligned}$$

This together with (7.2) implies

$$\int_M \phi |u|^q dv_g = \lim_{k \rightarrow \infty} \int_M \phi |u_k|^q dv_g = 1. \quad (7.3)$$

Since $u_k \rightharpoonup u$ weakly in E , we have

$$\int_M |\nabla u|^n dv_g = \lim_{k \rightarrow \infty} \int_M |\nabla u|^{n-2} \nabla u \nabla u_k dv_g \leq \limsup_{k \rightarrow \infty} \left(\int_M |\nabla u_k|^n dv_g \right)^{\frac{1}{n}} \left(\int_M |\nabla u|^n dv_g \right)^{1-\frac{1}{n}},$$

from which we obtain

$$\int_M |\nabla u|^n dv_g \leq \limsup_{k \rightarrow \infty} \int_M |\nabla u_k|^n dv_g. \quad (7.4)$$

In addition, we have by Fatou's lemma

$$\int_M v|u|^n dv_g \leq \limsup_{k \rightarrow \infty} \int_M v|u_k|^n dv_g. \quad (7.5)$$

Combining (7.3), (7.4) and (7.5), we conclude that S_q is attained by $u \in E \setminus \{0\}$. Since $|u| \in E$, one can easily see that S_q is also attained by $|u|$. \square

Now we get back to the problem (2.4). Since we are interested in nonnegative weak solutions, without loss of generality we may assume $f(x, t) \equiv 0$ for all $(x, t) \in M \times (-\infty, 0]$. By (f_1) , we have for all $(x, t) \in M \times \mathbb{R}$,

$$|F(x, t)| \leq \frac{b_1}{n} |t|^n + b_2 t \zeta\left(n, |t|^{\frac{n}{n-1}}\right).$$

This together with (ϕ_1) , (ϕ_2) and (2.2) implies that for any $u \in E$ there holds

$$\begin{aligned} \int_M \phi F(x, u) dv_g &\leq \|\phi\|_{L^p(B_O(R_0))} \|F(x, u)\|_{L^q(M)} + C_0 \int_M F(x, u) dv_g \\ &\leq \|\phi\|_{L^p(B_O(R_0))} \left(\frac{b_1}{n} \|u\|_{L^{qn}(M)}^n + b_2 \|u \zeta(n, |u|^{\frac{n}{n-1}})\|_{L^q(M)} \right) \\ &\quad + C_0 \frac{b_1}{n} \|u\|_{L^n(M)}^n + C_0 b_2 \|u \zeta(n, |u|^{\frac{n}{n-1}})\|_{L^1(M)} \\ &\leq C \left(\|u\|_{L^{qn}(M)}^n + \|u\|_{L^{qn}(M)} \left\| \zeta\left(n, \frac{qn}{n-1} |u|^{\frac{n}{n-1}}\right) \right\|_{L^1(M)}^{1-\frac{1}{n}} \right. \\ &\quad \left. \|u\|_{L^n(M)}^n + \|u\|_{L^n(M)} \left\| \zeta\left(n, \frac{n}{n-1} |u|^{\frac{n}{n-1}}\right) \right\|_{L^1(M)} \right), \end{aligned}$$

where C is a constant depending only on n , b_1 , b_2 , C_0 and $\|\phi\|_{L^p(B_O(R_0))}$, and $1/p + 1/q = 1$. By theorem 2.3, $u \in L^s(M)$ for all $s \geq n$, and for any $\alpha > 0$ there holds $\zeta(n, \alpha |u|^{\frac{n}{n-1}}) \in L^1(M)$. Hence

$$\int_M \phi F(x, u) dv_g < +\infty, \quad \forall u \in E.$$

Based on this, we can define a functional on E by

$$J(u) = \frac{1}{n} \|u\|_E^n - \int_M \phi F(x, u) dv_g. \quad (7.6)$$

By ([13], proposition 1) and the standard argument [34], we have $J \in C^1(E, \mathbb{R})$. Clearly the critical point of J is a weak solution to (2.4). Concerning the geometry of J , the following two lemmas imply that J has a mountain pass structure.

Lemma 7.3. *Assume that (f_1) , (f_2) , and (f_3) are satisfied. Then for any nonnegative, compactly supported function $u \in E \setminus \{0\}$, there holds $J(tu) \rightarrow -\infty$ as $t \rightarrow +\infty$.*

Proof. By (f_2) and (f_3) , there exist $c_1, c_2 > 0$ and $\mu > n$ such that $F(x, s) \geq c_1 s^\mu - c_2$ for all $(x, s) \in M \times [0, +\infty)$. Assume $\text{supp } u \subset B_O(R_1)$ for some $R_1 > 0$. We have

$$\begin{aligned} J(tu) &= \frac{t^n}{n} \|u\|_E^n - \int_{B_O(R_1)} \phi F(x, tu) dv_g \\ &\leq \frac{t^n}{n} \|u\|_E^n - c_1 t^\mu \int_{B_O(R_1)} \phi u^\mu dv_g - c_2 \int_{B_O(R_1)} \phi dv_g. \end{aligned}$$

This gives the desired result since $\phi(x) > 0$ for all $x \in M$ and $\mu > n$. \square

Lemma 7.4. *Assume that (f_1) and (f_4) are satisfied. Then there exist sufficiently small constants $r > 0$ and $\delta > 0$ such that $J(u) \geq \delta$ for all u with $\|u\|_E = r$.*

Proof. By (f_1) and (f_4) , there exists some constants $\theta \in (0, 1)$ and $C > 0$ such that

$$F(x, s) \leq \frac{(1-\theta)\lambda_\phi}{n} |s|^n + C|s|^{n+1} \zeta\left(n, \alpha_0 |s|^{\frac{n}{n-1}}\right)$$

for all $(x, s) \in M \times \mathbb{R}$. By definition of λ_ϕ ,

$$\frac{(1-\theta)\lambda_\phi}{n} \int_M \phi |u|^n dv_g \leq \frac{1-\theta}{n} \|u\|_E^n. \quad (7.7)$$

Note that ϕ satisfies (ϕ_1) and (ϕ_2) . We have by Hölder's inequality and (2.2) that

$$\begin{aligned} \int_M \phi |u|^{n+1} \zeta\left(n, \alpha_0 |u|^{\frac{n}{n-1}}\right) dv_g &\leq \|\phi\|_{L^p(B_O(R_0))} \left(\int_M |u|^{(n+1)q} dv_g \right)^{1/q} \left(\int_M \zeta\left(n, q' \alpha_0 |u|^{\frac{n}{n-1}}\right) dv_g \right)^{1/q'} \\ &\quad + C_0 \left(\int_M |u|^{(n+1)\beta} dv_g \right)^{1/\beta} \left(\int_M \zeta\left(n, \gamma \alpha_0 |u|^{\frac{n}{n-1}}\right) dv_g \right)^{1/\gamma}, \end{aligned} \quad (7.8)$$

where $1/p + 1/q + 1/q' = 1$ and $1/\beta + 1/\gamma = 1$. Fix $\alpha = \beta_0/2$, where β_0 is defined by (1.5). It follows from (i) of theorem 2.3 that there exists some constant τ depending only on α, n, K and i_0 such that

$$\Lambda_\alpha := \sup_{\|u\|_{1,\tau} \leq 1} \int_M \zeta\left(n, \alpha |u|^{\frac{n}{n-1}}\right) dv_g < +\infty. \quad (7.9)$$

Let r be a positive constant to be determined later. Now suppose $\|u\|_E = r$. It is easy to see that $\|u\|_{1,\tau} \leq r + \tau r / \nu_0^{1/n}$. Clearly one can select r sufficiently small such that $q' \alpha_0 \|u\|_{1,\tau}^{n/(n-1)} < \alpha$ and $\gamma \alpha_0 \|u\|_{1,\tau}^{n/(n-1)} < \alpha$. It follows from (7.9) that

$$\sup_{\|u\|_E = r} \int_M \zeta\left(n, q' \alpha_0 |u|^{\frac{n}{n-1}}\right) dv_g \leq \Lambda_\alpha$$

and

$$\sup_{\|u\|_E = r} \int_M \zeta\left(n, \gamma \alpha_0 |u|^{\frac{n}{n-1}}\right) dv_g \leq \Lambda_\alpha,$$

provided that r is chosen sufficiently small. Inserting these two inequalities into (7.8), then using the embedding $E \hookrightarrow L^s(M)$ for all $s \geq n$ (proposition 7.1) and (7.7), we obtain

$$J(u) \geq \frac{\theta}{n} \|u\|_E^n - \tilde{C} \|u\|_E^{n+1}$$

for some constant \tilde{C} depending only on α , n , K and i_0 , provided that $\|u\|_E$ is sufficiently small. This gives the desired result. \square

To estimate the min-max level of J , we state the following:

Lemma 7.5. *Assume (f_5) . There exists some nonnegative function $u^* \in E$ such that*

$$\sup_{t \geq 0} J(tu^*) < \frac{1}{n} \left(\frac{(p-1)\alpha_n}{p\alpha_0} \right)^{n-1}.$$

Proof. Let u^* be given by proposition 7.2, namely $u^* \geq 0$, $\|u^*\|_E = S_q$, and $\int_M \phi |u^*|^q dv_g = 1$. Then for any $t \geq 0$ there holds

$$\begin{aligned} J(tu^*) &= \frac{1}{n} \|tu^*\|_E^n - \int_M \phi(x) F(x, tu^*) dv_g \\ &\leq \frac{S_q^n}{n} t^n - \frac{C_q}{q} t^q \\ &\leq \frac{q-n}{nq} \frac{S_q^{nq/(q-n)}}{C_q^{n/(q-n)}} \\ &< \frac{1}{n} \left(\frac{(p-1)\alpha_n}{p\alpha_0} \right)^{n-1}. \end{aligned}$$

Here we have used the hypothesis (f_5) . \square

Adapting the proof of ([44], lemma 3.4), we obtain the following compactness result.

Lemma 7.6. *Assume (f_1) , (f_2) and (f_3) . Let $(u_j) \subset E$ be an arbitrary Palais-Smale sequence of J , i.e.,*

$$J(u_j) \rightarrow c, \quad J'(u_j) \rightarrow 0 \text{ in } E^* \text{ as } j \rightarrow \infty, \quad (7.10)$$

where E^ denotes the dual space of E . Then there exist a subsequence of (u_j) (still denoted by (u_j)) and $u \in E$ such that $u_j \rightharpoonup u$ weakly in E , $u_j \rightarrow u$ strongly in $L^q(M)$ for all $q \geq n$, and*

$$\begin{cases} \nabla u_j(x) \rightarrow \nabla u(x) & \text{a. e. in } M \\ \phi(x) F(x, u_j) \rightarrow \phi(x) F(x, u) & \text{strongly in } L^1(M). \end{cases}$$

Furthermore u is a weak solution of (2.4).

Proof. Assume (u_j) is a Palais-Smale sequence of J . By (7.10), we have

$$\frac{1}{n} \|u_j\|_E^n - \int_M \phi(x) F(x, u_j) dv_g \rightarrow c \text{ as } j \rightarrow \infty, \quad (7.11)$$

$$\left| \int_M (|\nabla_g u_j|^{n-2} \nabla_g u_j \nabla_g \psi + v |u_j|^{n-2} u_j \psi) dv_g - \int_M \phi(x) f(x, u_j) \psi dv_g \right| \leq \sigma_j \|\psi\|_E \quad (7.12)$$

for all $\psi \in E$, where $\sigma_j \rightarrow 0$ as $j \rightarrow \infty$. Note that $f(x, s) \equiv 0$ for all $(x, s) \in M \times (-\infty, 0]$. By (f_2) , we have $0 \leq \mu F(x, u_j) \leq u_j f(x, u_j)$ for some $\mu > n$. Taking $\psi = u_j$ in (7.12) and multiplying

(7.11) by μ , we have

$$\begin{aligned} \left(\frac{\mu}{n} - 1\right) \|u_j\|_E^n &\leq \mu|c| + \int_M \phi(x) (\mu F(x, u_j) - f(x, u_j)u_j) dv_g + \sigma_j \|u_j\|_E \\ &\leq \mu|c| + \sigma_j \|u_j\|_E. \end{aligned}$$

Therefore $\|u_j\|_E$ is bounded. It then follows from (7.11) and (7.12) that

$$\int_M \phi(x) f(x, u_j) u_j dv_g \leq C, \quad \int_M \phi(x) F(x, u_j) dv_g \leq C \quad (7.13)$$

for some constant C depending only on μ , n and c . By proposition 7.1, there exists some $u \in E$ such that $u_j \rightharpoonup u$ weakly in E , $u_j \rightarrow u$ strongly in $L^q(M)$ for any $q \geq n$, and $u_j \rightarrow u$ almost everywhere in M . By (f_3) , there exist positive constants A_1 and R_1 such that $F(x, s) \leq A_1 f(x, s)$ for all $s \geq R_1$. Particularly for any $A > R_1$ there holds

$$F(x, s) \leq A_1 f(x, s), \quad \forall s \geq A. \quad (7.14)$$

Now we prove that $\phi(x)F(x, u_j) \rightarrow \phi(x)F(x, u)$ strongly in $L^1(M)$. To this end, for any $\epsilon > 0$, we take $A > \max\{A_1 C / \epsilon, R_1\}$, where C is given by (7.13). Then we have by (7.14)

$$\int_{|u_j| > A} \phi(x) F(x, u_j) dv_g \leq \frac{A_1}{A} \int_M \phi(x) f(x, u_j) u_j dv_g < \epsilon. \quad (7.15)$$

In the same way

$$\int_{|u| > A} \phi(x) F(x, u) dv_g < \epsilon. \quad (7.16)$$

By (f_1) , we have for $(x, s) \in M \times [0, \infty)$

$$\begin{aligned} f(x, s) &\leq b_1 s^{n-1} + b_2 \zeta\left(n, \alpha_0 s^{\frac{n}{n-1}}\right) \\ &= b_1 s^{n-1} + b_2 s^n \sum_{k=n-1}^{\infty} \frac{\alpha_0^k s^{\frac{n}{n-1}(k-n+1)}}{k!} \\ &\leq b_1 s^{n-1} + b_2 s^n \alpha_0^{n-1} e^{\alpha_0 s^{\frac{n}{n-1}}}. \end{aligned}$$

Hence for all $(x, s) \in M \times [0, A]$ there holds

$$f(x, s) \leq \left(b_1 + b_2 \alpha_0^{n-1} A e^{\alpha_0 A^{\frac{n}{n-1}}}\right) s^{n-1}.$$

It follows that

$$F(x, s) \leq \frac{b_1 + b_2 \alpha_0^{n-1} A e^{\alpha_0 A^{\frac{n}{n-1}}}}{n} s^n, \quad \forall s \in [0, A].$$

for all $(x, s) \in M \times [0, A]$, which implies

$$|\phi(x) \chi_{\{|u_j| \leq A\}}(x) F(x, u_j)| \leq C_1 \phi(x) |u_j|^n, \quad (7.17)$$

where $C_1 = (b_1 + b_2\alpha_0^{n-1}Ae^{\alpha_0 A^{n/(n-1)}})/n$ and $\chi_{\{|u_j| \leq A\}}(x)$ denotes the characteristic function of the set $\{x \in M : |u_j(x)| \leq A\}$. By an inequality $\|a\|^n - \|b\|^n \leq n|a - b|(\|a\|^{n-1} + \|b\|^{n-1})$ ($\forall a, b \in \mathbb{R}$) and Hölder's inequality, we get

$$\begin{aligned} \int_M \phi \|u_j\|^n - \|u\|^n dv_g &\leq n \int_M \phi |u_j - u| (\|u_j\|^{n-1} + \|u\|^{n-1}) dv_g \\ &\leq n \left(\int_M \phi |u_j - u|^n dv_g \right)^{\frac{1}{n}} \left\{ \left(\int_M \phi \|u_j\|^n dv_g \right)^{1-\frac{1}{n}} + \left(\int_M \phi \|u\|^n dv_g \right)^{1-\frac{1}{n}} \right\}. \end{aligned}$$

Hence $\phi \|u_j\|^n \rightarrow \phi \|u\|^n$ in $L^1(M)$ since $u_j \rightarrow u$ strongly in $L^n(M)$. In view of (7.17), we conclude from the generalized Lebesgue's dominated convergence theorem

$$\lim_{j \rightarrow \infty} \int_M \phi(x) \chi_{\{|u_j| \leq A\}}(x) F(x, u_j) dv_g = \int_M \phi(x) \chi_{\{|u| \leq A\}}(x) F(x, u) dv_g.$$

This together with (7.15) and (7.16) implies that there exists some $m \in \mathbb{N}$ such that when $j > m$ there holds

$$\left| \int_M \phi F(x, u_j) dv_g - \int_M \phi F(x, u) dv_g \right| < 3\epsilon.$$

Therefore

$$\lim_{j \rightarrow \infty} \int_M \phi F(x, u_j) dv_g = \int_M \phi F(x, u) dv_g.$$

Using the same method as that of proving ([4], (4.26)), we have $\nabla_g u_j(x) \rightarrow \nabla_g u(x)$ for almost every $x \in M$ and

$$|\nabla_g u_j|^{n-2} \nabla_g u_j \rightharpoonup |\nabla_g u|^{n-2} \nabla_g u \quad \text{weakly in} \quad \left(L^{\frac{n}{n-1}}(M) \right)^n.$$

Passing to the limit $j \rightarrow \infty$ in (7.12), we obtain

$$\int_M (|\nabla_g u|^{n-2} \nabla_g u \nabla \psi + v |u|^{n-2} u \psi) dv_g - \int_M \phi(x) f(x, u) \psi dv_g = 0$$

for all $\psi \in C_0^\infty(M)$. Since $C_0^\infty(M)$ is dense in E under the norm $\|\cdot\|_E$, u is a weak solution of (2.4). \square

We say more words on lemma 7.6. Suppose (M, g) is the standard euclidian space \mathbb{R}^n and $\phi(x) = |x|^{-\beta}$, $0 \leq \beta < n$. The author [44] proved that $\phi F(x, u_j) \rightarrow \phi F(x, u)$ in $L^1(\mathbb{R}^n)$ under the assumption $E \hookrightarrow L^q(\mathbb{R}^n)$ compactly for all $q \geq 1$. While Lam-Lu [20] observed that the convergence still holds under the assumption $E \hookrightarrow L^q(\mathbb{R}^n)$ for all $q \geq n$. Here we generalized these two situations.

The following lemma is a nontrivial consequence of theorem 2.3. It is sufficient for our use when we consider the existence and multiplicity results for problems (2.4) and (2.8).

Lemma 7.7. *Let $(u_j) \subset E$ be any sequence of functions satisfying $\|u_j\|_E \leq 1$, $u_j \rightharpoonup u_0$ weakly in E , $\nabla_g u_j \rightarrow \nabla_g u_0$ almost everywhere in M , and $u_j \rightarrow u_0$ strongly in $L^n(M)$ as $j \rightarrow \infty$. Then*

(i) *for any $\alpha : 0 < \alpha < \alpha_n$, there holds*

$$\sup_j \int_M \zeta(n, \alpha |u_j|^{\frac{n}{n-1}}) dv_g < \infty; \tag{7.18}$$

(ii) for any $\alpha : 0 < \alpha < \alpha_n$ and $q : 0 < q < (1 - \|u_0\|_E^n)^{-1/(n-1)}$, there holds

$$\sup_j \int_M \zeta\left(n, q\alpha|u_j|^{\frac{n}{n-1}}\right) dv_g < \infty. \quad (7.19)$$

Proof. (i) For any fixed $\alpha : 0 < \alpha < \alpha_n$, it follows from part (i) of theorem 2.3 that there exists a positive constant τ_α depending only on α, n, K and i_0 such that

$$\mathcal{B}_\alpha = \sup_{u \in W^{1,n}(M), \|u\|_{1,\tau_\alpha} \leq 1} \int_M \zeta\left(n, \alpha|u|^{\frac{n}{n-1}}\right) dv_g < \infty. \quad (7.20)$$

Note that $v \geq v_0$ in M . Since $\|u_j\|_E \leq 1$, we get

$$\|u_j\|_{1,\tau_\alpha} = \left(\int_M |\nabla_g u_j|^n dv_g \right)^{\frac{1}{n}} + \tau_\alpha \left(\int_M |u_j|^n dv_g \right)^{\frac{1}{n}} \leq 1 + \frac{\tau_\alpha}{v_0^{1/n}}.$$

There exists some small positive number α_0 such that $\alpha_0 \|u_j\|_{1,\tau_\alpha}^{\frac{n}{n-1}} \leq \alpha$. Hence by (7.20), there holds

$$\sup_j \int_M \zeta\left(n, \alpha_0|u_j|^{\frac{n}{n-1}}\right) dv_g \leq \sup_j \int_M \zeta\left(n, \alpha \left| \frac{u_j}{\|u_j\|_{1,\tau_\alpha}} \right|^{\frac{n}{n-1}}\right) dv_g \leq \mathcal{B}_\alpha.$$

This allows us to define

$$\alpha^* = \sup \left\{ \alpha : \sup_j \int_M \zeta\left(n, \alpha|u_j|^{\frac{n}{n-1}}\right) dv_g < \infty \right\}.$$

To prove (7.18), it suffices to prove that $\alpha^* \geq \alpha_n$. Suppose not, we have $\alpha^* < \alpha_n$. Take two constants α' and α'' such that $\alpha^* < \alpha' < \alpha'' < \alpha_n$. By part (i) of theorem 2.3 again, there exists some constant $\tau_{\alpha''}$ depending only on α'', n, K and i_0 such that

$$\mathcal{B}_{\alpha''} = \sup_{u \in W^{1,n}(M), \|u\|_{1,\tau_{\alpha''}} \leq 1} \int_M \zeta\left(n, \alpha''|u|^{\frac{n}{n-1}}\right) dv_g < \infty. \quad (7.21)$$

Since $u_j \rightarrow u_0$ strongly in $L^n(M)$ and $\nabla_g u_j \rightarrow \nabla_g u_0$ a. e. in M , we obtain by using Brezis-Lieb's lemma [6]

$$\|u_j - u_0\|_{1,\tau_{\alpha''}} = \left(\int_M |\nabla_g u_j|^n dv_g - \int_M |\nabla_g u_0|^n dv_g \right)^{1/n} + o_j(1),$$

where $o_j(1) \rightarrow 0$ as $j \rightarrow \infty$. Since $u_j \rightharpoonup u_0$ weakly in E , there holds

$$\lim_{j \rightarrow +\infty} \int_M |\nabla_g u_0|^{n-2} \nabla_g u_0 \nabla_g u_j dv_g = \int_M |\nabla_g u_0|^n dv_g.$$

This immediately implies that

$$\int_M |\nabla_g u_0|^n dv_g \leq \limsup_{j \rightarrow +\infty} \int_M |\nabla_g u_j|^n dv_g \leq 1.$$

Hence

$$\|u_j - u_0\|_{1,\tau_{\alpha''}} \leq 1 + o_j(1).$$

It follows from (2.3) that for any $\epsilon > 0$ there exists some constant \tilde{c} depending only on ϵ and n such that

$$\zeta\left(n, \alpha'|u_j|^{\frac{n}{n-1}}\right) \leq \frac{1}{\mu}\zeta\left(n, \alpha'(1+\epsilon)\mu|u_j - u_0|^{\frac{n}{n-1}}\right) + \frac{1}{\nu}\zeta\left(n, \alpha'\tilde{c}\nu|u_0|^{\frac{n}{n-1}}\right), \quad (7.22)$$

where $1/\mu + 1/\nu = 1$. Choosing ϵ sufficiently small and μ sufficiently close to 1 such that

$$\alpha'(1+\epsilon)\mu\|u_j - u_0\|_{1, \tau_{\alpha''}}^{\frac{n}{n-1}} < \alpha'',$$

provided that j is sufficiently large. This together with (7.21) implies that

$$\sup_j \int_M \zeta\left(n, \alpha'(1+\epsilon)\mu|u_j - u_0|^{\frac{n}{n-1}}\right) dv_g \leq \mathcal{B}_{\alpha''}. \quad (7.23)$$

In addition, we have by part (iii) of theorem 2.3 that

$$\int_M \zeta\left(n, \alpha'\tilde{c}\nu|u_0|^{\frac{n}{n-1}}\right) dv_g < +\infty. \quad (7.24)$$

Inserting (7.23) and (7.24) into (7.22), we get

$$\sup_j \int_M \zeta\left(n, \alpha'|u_j|^{\frac{n}{n-1}}\right) dv_g < +\infty,$$

which contradicts the definition of α^* and thus ends the proof of part (i).

(ii) Given any $\alpha : 0 < \alpha < \alpha_n$ and any $q : 0 < q < (1 - \|u_0\|_E^n)^{-1/(n-1)}$. By (2.3), $\forall \epsilon > 0$, there exist constants $\tilde{c} > 0$, $\mu > 1$ and $\nu > 1$ ($1/\mu + 1/\nu = 1$) such that

$$\int_M \zeta\left(n, q\alpha|u_j|^{\frac{n}{n-1}}\right) dv_g \leq \frac{1}{\mu} \int_M \zeta\left(n, q\alpha(1+\epsilon)\mu|u_j - u_0|^{\frac{n}{n-1}}\right) dv_g + \frac{1}{\nu} \int_M \zeta\left(n, q\alpha\tilde{c}\nu|u_0|^{\frac{n}{n-1}}\right) dv_g.$$

By Brezis-Lieb's lemma [6],

$$\|u_j - u_0\|_E^{\frac{n}{n-1}} \leq (1 - \|u_0\|_E^n)^{\frac{1}{n-1}} + o_j(1).$$

If we choose ϵ sufficiently small and μ sufficiently close to 1 such that

$$q\alpha(1+\epsilon)\mu\|u_j - u_0\|_E^{\frac{n}{n-1}} \leq (\alpha + \alpha_n)/2,$$

provided that j is sufficiently large. It then follows from part (i) that

$$\sup_j \int_M \zeta\left(n, q\alpha(1+\epsilon)\mu|u_j - u_0|^{\frac{n}{n-1}}\right) dv_g < +\infty.$$

By part (iii) of theorem 2.3, we have

$$\int_M \zeta\left(n, q\alpha\tilde{c}\nu|u_0|^{\frac{n}{n-1}}\right) dv_g < +\infty.$$

Therefore (7.19) holds. □

Remark 7.8. In lemma 7.7, if $u_0 \equiv 0$, then the conclusion of (ii) is weaker than that of (i). If $u_0 \not\equiv 0$, then the conclusion of (i) is a special case of that of (ii). If (M, g) has dimension two, the assumption $\nabla_g u_j \rightarrow \nabla_g u_0$ almost everywhere in M can be removed.

Proof of theorem 2.7. It follows from lemma 7.3 and lemma 7.4 that J satisfies all the hypothesis of the mountain-pass theorem except for the Palais-Smale condition: $J \in C^1(E, \mathbb{R})$; $J(0) = 0$; $J(u) \geq \delta > 0$ when $\|u\|_E = r$; $J(e) < 0$ for some $e \in E$ with $\|e\|_E > r$. Then using the mountain-pass theorem without the Palais-Smale condition [34], we can find a sequence (u_j) in E such that

$$J(u_j) \rightarrow c > 0, \quad J'(u_j) \rightarrow 0 \text{ in } E^*,$$

where

$$c = \min_{\gamma \in \Gamma} \max_{u \in \gamma} J(u) \geq \delta$$

is the min-max value of J , where $\Gamma = \{\gamma \in C([0, 1], E) : \gamma(0) = 0, \gamma(1) = e\}$. This is equivalent to (7.11) and (7.12). By lemma 7.6, up to a subsequence, there holds

$$\begin{cases} u_j \rightharpoonup u \text{ weakly in } E \\ u_j \rightarrow u \text{ strongly in } L^q(M), \quad \forall q \geq n \\ \lim_{j \rightarrow \infty} \int_M \phi(x) F(x, u_j) dv_g = \int_M \phi(x) F(x, u) dv_g \\ u \text{ is a weak solution of (2.4).} \end{cases} \quad (7.25)$$

Now suppose by contradiction $u \equiv 0$. Since $F(x, 0) = 0$ for all $x \in M$, it follows from (7.11) and (7.25) that

$$\lim_{j \rightarrow \infty} \|u_j\|_E^n = nc > 0. \quad (7.26)$$

By lemma 7.5, $0 < c < \frac{1}{n} \left(\frac{(p-1)\alpha_n}{p\alpha_0} \right)^{n-1}$. Thus there exists some $\eta_0 > 0$ and $m > 0$ such that $\|u_j\|_E^n \leq \left(\frac{p-1}{p} \frac{\alpha_n}{\alpha_0} - \eta_0 \right)^{n-1}$ for all $j > m$. Choose $q > 1$ sufficiently close to 1 such that $q\alpha_0 \|u_j\|_E^{\frac{n}{n-1}} \leq (1 - 1/p)\alpha_n - \alpha_0\eta_0/2$ for all $j > m$. By (f₁),

$$|f(x, u_j)u_j| \leq b_1 |u_j|^n + b_2 |u_j| \zeta \left(n, \alpha_0 |u_j|^{\frac{n}{n-1}} \right).$$

It follows from (2.2), Hölder's inequality, and part (i) of lemma 7.7 that

$$\begin{aligned} \int_M \phi |f(x, u_j)u_j| dv_g &\leq b_1 \int_M \phi |u_j|^n dv_g + b_2 \int_M \phi |u_j| \zeta \left(n, \alpha_0 |u_j|^{\frac{n}{n-1}} \right) dv_g \\ &\leq b_1 \int_M \phi |u_j|^n dv_g + b_2 \left(\int_M \phi |u_j|^{q'} dv_g \right)^{1/q'} \left(\int_M \phi \zeta \left(n, q\alpha_0 |u_j|^{\frac{n}{n-1}} \right) dv_g \right)^{1/q} \\ &\leq b_1 \int_M \phi |u_j|^n dv_g + C \left(\int_M \phi |u_j|^{q'} dv_g \right)^{1/q'} \rightarrow 0 \quad \text{as } j \rightarrow \infty, \end{aligned}$$

where $1/q + 1/q' = 1$ and C is some constant which is independent of j . Here we have used (7.25) again (precisely $u_j \rightarrow u$ in $L^r(\mathbb{R}^N)$ for all $r \geq n$) in the above estimates. Inserting this into (7.12) with $\psi = u_j$, we have

$$\|u_j\|_E \rightarrow 0 \quad \text{as } j \rightarrow \infty,$$

which contradicts (7.26). Therefore $u \not\equiv 0$ and we obtain a nontrivial weak solution of (2.4). Finally u is nonnegative since $f(x, s) \equiv 0$ for all $(x, s) \in M \times (-\infty, 0]$. \square

Proof of theorem 2.10. Since the proof is very similar to that of ([44], theorem 1.2), we only give its sketch and emphasize the difference between these two situations. Instead of $J : E \rightarrow \mathbb{R}$ defined by (7.6), we consider functionals for all $u \in E$ and $\epsilon > 0$

$$J_\epsilon(u) = \frac{1}{n} \|u\|_E^n - \int_M \phi(x) F(x, u) dv_g - \epsilon \int_M h u dv_g.$$

Firstly, lemma 7.6 still holds if we replace J by J_ϵ . Namely for any Palais-Smale sequence $(u_j) \subset E$ of J_ϵ , there exist a subsequence of (u_j) (still denoted by (u_j)) and $u \in E$ such that $u_j \rightharpoonup u$ weakly in E , $u_j \rightarrow u$ strongly in $L^q(M)$ for all $q \geq n$, and

$$\begin{cases} \nabla_g u_j(x) \rightarrow \nabla_g u(x) & \text{a. e. in } M \\ \phi(x) F(x, u_j) \rightarrow \phi(x) F(x, u) & \text{strongly in } L^1(M) \\ u & \text{is a weak solution of (2.8).} \end{cases} \quad (7.27)$$

Secondly, using the same method in the first two steps of the proof of ([44], theorem 1.2), we have the following:

(a) there exist constants $\epsilon_1 > 0, \delta > 0$, and a sequence of functions $(v_j) \subset E$ such that $J_\epsilon(v_j) \rightarrow c_M$ and $J'_\epsilon(v_j) \rightarrow 0$ as $j \rightarrow \infty$, provided that $0 < \epsilon < \epsilon_1$. In addition, v_j is bounded in E , $v_j \rightharpoonup u_M$ weakly in E and u_M is a weak solution of (2.8). Here c_M is the min-max value of J_ϵ and satisfies

$$0 < c_M < \frac{1}{n} \left(1 - \frac{1}{p}\right)^{n-1} \left(\frac{\alpha_n}{\alpha_0}\right)^{n-1} - \delta; \quad (7.28)$$

(b) there exists a constant $\epsilon_2 : 0 < \epsilon_2 < \epsilon_1$ such that for any $\epsilon : 0 < \epsilon < \epsilon_2$, there exist positive constant r_ϵ with $r_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ and sequence $(u_j) \subset E$ such that

$$J_\epsilon(u_j) \rightarrow c_\epsilon = \inf_{\|u\|_E \leq r_\epsilon} J_\epsilon(u) < 0, \quad J'_\epsilon(u_j) \rightarrow 0 \quad \text{in } E^* \quad \text{as } j \rightarrow \infty.$$

In addition, $u_j \rightarrow u_0$ strongly in E , where u_0 is a weak solution of (2.8) with $J_\epsilon(u_0) = c_\epsilon$.

Thirdly, there exists $\epsilon_0 : 0 < \epsilon_0 < \epsilon_2$ such that if $0 < \epsilon < \epsilon_0$, then $u_M \not\equiv u_0$. Suppose by contradiction that $u_M \equiv u_0$. Then $v_j \rightharpoonup u_0$ weakly in E . By (a),

$$J_\epsilon(v_j) \rightarrow c_M > 0, \quad |\langle J'_\epsilon(v_j), \varphi \rangle| \leq \gamma_j \|\varphi\|_E \quad (7.29)$$

with $\gamma_j \rightarrow 0$ as $j \rightarrow \infty$. On one hand we have by (7.27),

$$\int_M \phi(x) F(x, v_j) dv_g \rightarrow \int_M \phi(x) F(x, u_0) dv_g \quad \text{as } j \rightarrow \infty. \quad (7.30)$$

On the other hand, since $v_j \rightharpoonup u_0$ weakly in E and $h \in E^*$, it follows that

$$\int_M h v_j dv_g \rightarrow \int_M h u_0 dv_g \quad \text{as } j \rightarrow \infty. \quad (7.31)$$

Inserting (7.30) and (7.31) into the first equality of (7.29), we obtain

$$\frac{1}{n}\|v_j\|_E^n = c_M + \int_M \phi(x)F(x, u_0)dv_g + \epsilon \int_M hu_0dv_g + o_j(1). \quad (7.32)$$

In the same way, one can derive

$$\frac{1}{n}\|u_j\|_E^n = c_\epsilon + \int_M \phi(x)F(x, u_0)dv_g + \epsilon \int_M hu_0dv_g + o_j(1). \quad (7.33)$$

Combining (7.32) and (7.33), we have

$$\|v_j\|_E^n - \|u_0\|_E^n = n(c_M - c_\epsilon + o_j(1)). \quad (7.34)$$

From (b), we know that $c_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$. This together with (7.28) leads to the existence of $\epsilon_0 : 0 < \epsilon_0 < \epsilon_2$ such that if $0 < \epsilon < \epsilon_0$, then

$$0 < c_M - c_\epsilon < \frac{1}{n} \left(\frac{p-1}{p} \frac{\alpha_n}{\alpha_0} \right)^{n-1}. \quad (7.35)$$

Write

$$w_j = \frac{v_j}{\|v_j\|_E}, \quad w_0 = \frac{u_0}{(\|u_0\|_E^n + n(c_M - c_\epsilon))^{1/n}}.$$

It follows from (7.34) and $v_j \rightharpoonup u_0$ weakly in E that $w_j \rightharpoonup w_0$ weakly in E . Note that

$$\int_M \phi(x)\zeta(n, \alpha_0|v_j|^{n/(n-1)})dv_g = \int_M \phi(x)\zeta(n, \alpha_0\|v_j\|_E^{n/(n-1)}|w_j|^{n/(n-1)})dv_g.$$

By (7.34) and (7.35), a straightforward calculation shows

$$\lim_{j \rightarrow \infty} \alpha_0\|v_j\|_E^{\frac{n}{n-1}} (1 - \|w_0\|_E^n)^{\frac{1}{n-1}} < \left(1 - \frac{1}{p}\right) \alpha_n.$$

Hence lemma 7.7 together with (2.3) implies that $\phi(x)\zeta(n, \alpha_0|v_j|^{n/(n-1)})$ is bounded in $L^q(M)$ for some $q : 1 < q < n/(n-1)$. By (f_1) ,

$$|f(x, v_j)| \leq b_1|v_j|^{n-1} + b_2\zeta(n, \alpha_0|v_j|^{\frac{n}{n-1}}).$$

By the definition of ζ there exists a constant $c > 0$ such that

$$|f(x, v_j)\chi_{\{|v_j| \leq 1\}}(x)| \leq c|v_j|^{n-1}, \quad |f(x, v_j)\chi_{\{|v_j| > 1\}}(x)| \leq c\zeta(n, \alpha_0|v_j|^{\frac{n}{n-1}}),$$

where χ_B denotes the characteristic function of $B \subset M$. Hence

$$\begin{aligned} \left| \int_M \phi(x)f(x, v_j)(v_j - u_0)dv_g \right| &\leq c \int_M \phi(x) \left(|v_j|^{n-1} + \zeta(n, \alpha_0|v_j|^{\frac{n}{n-1}}) \right) |v_j - u_0|dv_g \\ &\leq c \left\| \phi|v_j|^{n-1} \right\|_{L^{\frac{n}{n-1}}(M)} \|v_j - u_0\|_{L^n(M)} \\ &\quad + c \left\| \phi\zeta(n, \alpha_0|v_j|^{\frac{n}{n-1}}) \right\|_{L^q(M)} \|v_j - u_0\|_{L^{q'}(M)}. \end{aligned}$$

Since $1 < q < n/(n-1)$, we have $q' > n$. Then it follows from the compact embedding $E \hookrightarrow L^r(M)$ for all $r \geq n$ that

$$\lim_{j \rightarrow \infty} \int_M \phi(x) f(x, v_j)(v_j - u_0) dv_g = 0. \quad (7.36)$$

Taking $\varphi = v_j - u_0$ in (7.29), we have by using (7.31) and (7.36) that

$$\int_M (|\nabla_g v_j|^{n-2} \nabla_g v_j \nabla_g (v_j - u_0) + v(x) |v_j|^{n-2} v_j (v_j - u_0)) dv_g \rightarrow 0. \quad (7.37)$$

However the fact $v_n \rightharpoonup u_0$ weakly in E leads to

$$\int_M (|\nabla_g u_0|^{n-2} \nabla_g u_0 \nabla_g (v_j - u_0) + v(x) |u_0|^{n-2} u_0 (v_j - u_0)) dv_g \rightarrow 0. \quad (7.38)$$

Subtracting (7.38) from (7.37), using the well known inequality (see [26], chapter 10)

$$2^{n-1} |b - a|^n \leq \langle |b|^{n-2} b - |a|^{n-2} a, b - a \rangle, \quad \forall a, b \in \mathbb{R}^n,$$

we have $\|v_j - u_0\|_E^n \rightarrow 0$ as $j \rightarrow \infty$. This together with (7.34) implies that $c_M = c_\epsilon$, which is absurd since $c_M > 0$ and $c_\epsilon < 0$. Therefore $u_M \not\equiv u_0$. Since $f(x, s) \equiv 0$ for all $(x, s) \in M \times (-\infty, 0]$, one can easily see that $u_M \geq 0$ and $u_0 \geq 0$. This completes the proof of the theorem. \square

Finally we shall construct examples of f 's to show that (f_1) -(f_5) do not imply (H_5) .

Proof of proposition 2.9. Let ϕ satisfies the hypotheses (ϕ_1) and (ϕ_2) , $p > 1$ be given in (ϕ_1) , l be an integer satisfying $l \geq n$, $q = nl/(n-1) + 1$ and S_q be defined by (2.7). In view of lemma 7.2, S_q is attained by some nonnegative function $u \in E$. Let C_q be a positive number such that

$$C_q > \left(\frac{q-n}{q} \right)^{(q-n)/n} \left(\frac{p\alpha_0}{(p-1)\alpha_n} \right)^{(q-n)(n-1)/n} S_q^q.$$

Let $\chi : [0, \infty) \rightarrow \mathbb{R}$ be a smooth function such that $0 \leq \chi \leq 1$, $\chi \equiv 0$ on $[0, A]$, $\chi \equiv 1$ on $[2A, \infty)$, and $|\chi'| \leq 2/A$, where A is a positive constant to be determined later. We set

$$f(t) = \begin{cases} 2^l l! C_q \sum_{k=l}^{\infty} \frac{(t^{\frac{n}{n-1}} - \chi(t) t^{\frac{1}{n-1}})^k}{k!}, & t \geq 0 \\ 0, & t < 0. \end{cases}$$

Now we check (f_1) -(f_5) for appropriate choice of A as follows.

(f_1) : If $A > 1$, then $0 \leq t^{n/(n-1)} - \chi(t) t^{1/(n-1)} \leq t^{n/(n-1)}$ for all $t \geq 0$. Thus

$$\begin{aligned} f(t) &= 2^l l! C_q \left(e^{t^{n/(n-1)} - \chi(t) t^{1/(n-1)}} - \sum_{k=0}^{l-1} \frac{(t^{\frac{n}{n-1}} - \chi(t) t^{\frac{1}{n-1}})^k}{k!} \right) \\ &\leq 2^l l! C_q \left(e^{t^{n/(n-1)}} - \sum_{k=0}^{l-1} \frac{t^{\frac{nk}{n-1}}}{k!} \right) \\ &\leq 2^l l! C_q \zeta(n, t^{n/(n-1)}) \end{aligned}$$

for all $t \geq 0$. So (f_1) is satisfied when $A > 1$.

(f_2) : When $t \in [0, A]$, we have $\chi(t) = 0$ and

$$\int_0^t f(t)dt = 2^l l! C_q \sum_{k=l}^{\infty} \int_0^t \frac{t^{\frac{nk}{n-1}}}{k!} dt \leq 2^l l! C_q t \sum_{k=l}^{\infty} \frac{t^{\frac{nk}{n-1}}}{k!} = t f(t). \quad (7.39)$$

When $t \geq A$, we *claim* that if A is chosen sufficiently large, say $A \geq 4^{n-1}$, then

$$\int_A^t \frac{(t^{\frac{n}{n-1}} - \chi(t)t^{\frac{1}{n-1}})^k}{k!} dt \leq \frac{(t^{\frac{n}{n-1}} - \chi(t)t^{\frac{1}{n-1}})^{k+1}}{(k+1)!} - \frac{A^{\frac{n(k+1)}{n-1}}}{(k+1)!}. \quad (7.40)$$

In fact, if we set

$$\gamma(t) = \int_A^t \frac{(t^{\frac{n}{n-1}} - \chi(t)t^{\frac{1}{n-1}})^k}{k!} dt - \frac{(t^{\frac{n}{n-1}} - \chi(t)t^{\frac{1}{n-1}})^{k+1}}{(k+1)!} + \frac{A^{\frac{n(k+1)}{n-1}}}{(k+1)!},$$

then $\gamma(A) = 0$ and

$$\gamma'(t) = \frac{(t^{\frac{n}{n-1}} - \chi(t)t^{\frac{1}{n-1}})^k}{k!} - \frac{(t^{\frac{n}{n-1}} - \chi(t)t^{\frac{1}{n-1}})^k}{k!} \left(\frac{n}{n-1} t^{\frac{1}{n-1}} - \chi'(t)t^{\frac{1}{n-1}} - \frac{1}{n-1} \chi(t)t^{\frac{1}{n-1}-1} \right).$$

Let $A \geq 4^{n-1}$. Then for $t \in [A, \infty)$ there holds

$$\begin{aligned} \frac{n}{n-1} t^{\frac{1}{n-1}} - \chi'(t)t^{\frac{1}{n-1}} - \frac{1}{n-1} \chi(t)t^{\frac{1}{n-1}-1} &\geq \left(\frac{n}{n-1} - \frac{2}{A} \right) A^{\frac{1}{n-1}} - \frac{1}{n-1} A^{\frac{1}{n-1}-1} \\ &\geq 4 \left(\frac{n}{n-1} - \frac{2}{4(n-1)} - \frac{1}{4(n-1)^2} \right) \\ &> 1. \end{aligned}$$

Hence $\gamma'(t) \leq 0$ and thus our claim (7.40) holds. Note that

$$\int_0^A \frac{t^{\frac{nk}{n-1}}}{k!} dt = \frac{A^{\frac{n(k+1)}{n-1}}}{(k+1)!} \frac{k+1}{\frac{nk}{n-1} + 1} A^{-\frac{1}{n-1}} \leq \frac{A^{\frac{n(k+1)}{n-1}}}{(k+1)!}. \quad (7.41)$$

It follows from (7.40) and (7.41) that when $t \geq A$,

$$\int_0^t \frac{(t^{\frac{n}{n-1}} - \chi(t)t^{\frac{1}{n-1}})^k}{k!} dt \leq \frac{(t^{\frac{n}{n-1}} - \chi(t)t^{\frac{1}{n-1}})^{k+1}}{(k+1)!},$$

and whence

$$\int_0^t f(t)dt \leq f(t) \leq \frac{1}{\mu} t f(t) \quad (7.42)$$

for some $\mu > n$. This together with (7.39) implies that (f_2) holds.

(f_3) : Let $A \geq 4^{n-1}$. In view of (7.42), when $t \geq A$,

$$F(t) = \int_0^t f(t)dt \leq f(t).$$

Hence (f_3) is satisfied.

(f_4) : Since $l > n$, we get $F(t)/t^n \rightarrow 0$ as $t \rightarrow 0+$. Hence (f_4) holds.

(f_5) : Note that $t^{n/(n-1)} - t^{1/(n-1)} \geq t^{n/(n-1)}/2$ for all $t \geq 2$. Let $A \geq 2$. Then for all $t \geq A$ there holds

$$f(t) \geq 2^l l! C_q \frac{(t^{\frac{n}{n-1}} - t^{\frac{1}{n-1}})^l}{l!} \geq 2^l C_q (t^{\frac{n}{n-1}}/2)^l = C_q t^{q-1}.$$

When $t \in [0, A]$, we get

$$f(t) \geq 2^l l! C_q \frac{t^{\frac{n}{n-1}}}{l!} = 2^l C_q t^{q-1}.$$

Hence (f_5) is satisfied. In short, $f(t)$ satisfies (f_1) – (f_5) if $A \geq 4^{n-1}$.

Finally we check that (H_5) does not hold. When $t \geq 2A$, we have

$$f(t) = 2^l l! C_q \left(e^{t^{\frac{n}{n-1}} - t^{\frac{1}{n-1}}} - \sum_{k=0}^{l-1} \frac{(t^{\frac{n}{n-1}} - t^{\frac{1}{n-1}})^k}{k!} \right).$$

It follows that

$$\lim_{t \rightarrow +\infty} t f(t) e^{-t^{\frac{n}{n-1}}} = 0.$$

Thus $f(t)$ does not satisfy (H_5) . □

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